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Unit Group of the Group Algebra $\mathbb{F}_q GL(2,7)$

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Abstract. In this paper, we consider the general linear group $GL(2,7)$ of 2×2 invertible matrices over the finite field of order 7 and compute the unit group of the semisimple group algebra $\mathbb{F}_q GL(2,7)$, where $\mathbb{F}_q$ is a finite field. For the computation of the unit group, we need the Wedderburn decomposition of $\mathbb{F}_q GL(2,7)$, which is determined by first computing the Wedderburn decomposition of the group algebra $\mathbb{F}_q(PSL(3,2) \rtimes C_2)$. Here $PSL(3,2)$ is the projective special linear group of degree 3 over a finite field of 2 elements.

Key Words: Unit Group, Group Algebra, General Linear Group, Wedderburn Decomposition
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1 Introduction

Let $\mathcal{U}(\mathbb{F}_q G)$ be the group of units of the group algebra $\mathbb{F}_q G$ of the finite group G over the finite field $\mathbb{F}_q$ of cardinality $q = p^k$, where p is the characteristic of $\mathbb{F}_q$ and $k \geq 1$. The structure of unit group of the group algebra has inevitably developed into a significant area of research owing to the applications of the units in many fields, including convolutional codes [13], cryptography [14,19], etc. In addition to this, the recent counterexample to the renowned *Kaplansky's unit conjecture* further emphasizes the need of research in this area (see [11]). It has been extensively investigated how the unit group of the group algebra $\mathbb{F}_q G$ is structured (see [1,3,4,16–18,20,21,23,25,27,28]). Furthermore, there have been significant developments in the exploration of the unit group of modular group algebras, in addition to integral and semisimple group algebras (see [5–8] and the references therein for a comprehensive and recent literature in this direction). One of the most important studies in this area examines the unit groups of the semisimple group algebras of all metabelian groups (see [4]). As a result, the majority of research in this area concentrates on the comprehension of the unit group of non-metabelian group algebras. We recall that a group is non-metabelian if its derived subgroup is non-abelian.

As of order 120, Mittal et al. [20] described the unit groups of the group algebras of non-metabelian groups. Further, Sharma et al. [15,24] identified the unit group of the semisimple group algebra for the special linear groups $SL(2,3)$ and $SL(2,5)$. Sivaranjani et al. [27] studied the unit group of the semisimple group algebra for the special linear groups $SL(2,8)$ and $SL(2,9)$. The unit group of the semisimple group algebra for the general linear group $GL(2,3)$ is studied in [20]. Further, we note that the groups $GL(2,\mathbb{Z}_4)$ and $GL(2,6)$ are, respectively, isomorphic to the direct product of A_5 and C_3 and direct product of $GL(2,3)$ and S_3. Consequently, the unit group of their associated group algebras can be easily characterized by using the tensor product formula (see [22]). Furthermore, Sivaranjani et al. [26] determined the unit group of the semisimple group algebra of the group $GL(2,5)$.

In this paper, in continuation with the previous works, we determine the unit group of the semisimple group algebra of the group $GL(2,7)$. As usual, to compute the unit group, we study the Wedderburn decomposition of the group algebra $\mathbb{F}_qGL(2,7)$. It is important to note that we need the Wedderburn decomposition of the group algebra $\mathbb{F}_q(PSL(3,2) \rtimes C_2)$ for computing the Wedderburn decomposition of $\mathbb{F}_qGL(2,7)$. Here $PSL(3,2)$ is the projective special linear group of degree 3 over a finite field of 2 elements. Finally, the unit group can be easily derived from the Wedderburn decomposition.

The paper is organized as follows. Section 2 covers the prerequisites. The main results of the paper appear in Section 3. Section 4 concludes the paper.

2 Preliminaries

Throughout this paper, $GL(n,p)$ denotes the general linear group of $n \times n$ invertible matrices over the field $\mathbb{Z}_p$, where p is a prime. The order of the general linear group $GL(n,p)$ is given by

$$(p^n - 1)(p^n - p)\cdots(p^n - p^{n-1}). \tag{1}$$

It follows from (1) that the order of $GL(2,7)$ is 2016.

Next, we discuss some notations and results from [9]. Let $J(\mathbb{F}_qG)$ denote the Jacobson radical of $\mathbb{F}_qG$. Let s be the exponent of the group G, i.e., the l.c.m of the order of p regular element, and let η be the primitive s^{th} root of unity over a finite field $\mathbb{F}$. Put

$$T_{G,\mathbb{F}} = \{t \in \mathbb{Z}^+ : \eta \to \eta^t \text{ is an automorphism of } \mathbb{F}(\eta) \text{ over } \mathbb{F}\}.$$

Since the Galois group $\mathrm{Gal}(\mathbb{F}(\eta) : \mathbb{F})$ is cyclic, for any $\sigma \in \mathrm{Gal}(\mathbb{F}(\eta) : \mathbb{F})$, there exists a postive integer s such that $\sigma(\eta) = \eta^s$. For any p-regular element $g \in G$ (i.e., p does not divide order of g), we define $\gamma_g = \sum h$, where h runs over all the elements in the conjugacy class C_g of g. The cyclotomic $\mathbb{F}$-class of γ_g is defined as $S\mathbb{F}(\gamma_g) = \{\gamma_{g^t} \mid t \in T_{G,\mathbb{F}}\}$.

The following theorem characterizes the set $T_{G,\mathbb{F}}$.

Theorem 2.1 *[20, Theorem 2.3] Let $\mathbb{F}$ be a finite field with prime power order d such that $\gcd(d,s) = 1$ and $e = order_s(d)$ is the multiplicative order of d modulo s. Then $T_{G,\mathbb{F}} = \{1, d, ..., d^{e-1}\} \mod s$.*

To uniquely identify the Wedderburn decomposition (WD) of the group algebra, the following six results will play an important role.

Proposition 2.1 *[9, Proposition 1.2] The number of non-isomorphic simple components of $\mathbb{F}G/J(\mathbb{F}G)$ is equal to the number of cyclotomic $\mathbb{F}$-classes in G.*

Theorem 2.2 *[9, Theorem 1.3] Assume G has t cyclotomic $\mathbb{F}$-classes and $Gal(\mathbb{F}(\eta) : \mathbb{F}))$ is a cyclic group. Then $|S_i| = [\mathbb{F}_i : \mathbb{F}]$ with appropriate index ordering if $S_1, S_2, \cdots, S_t$ are the cyclotomic $\mathbb{F}$-classes of G and $\mathbb{F}_1, \mathbb{F}_2, \cdots, \mathbb{F}_t$ are the simple components of $Z(\mathbb{F}G/J(\mathbb{F}G))$.*

Proposition 2.2 *[18, Proposition 3.6.11] Let G' be the commutator subgroup of G and let $\mathbb{F}G$ be a semisimple group algebra. Then*

$$\mathbb{F}G \simeq \mathbb{F}(G/G') \oplus \triangle(G, G'),$$

where $\mathbb{F}(G/G')$ is the sum of all commutative simple components of $\mathbb{F}G$ and $\triangle(G, G')$ is the sum of all others.

Proposition 2.3 *[18, Proposition 3.6.7] Let N be a normal subgroup of G and let $\mathbb{F}G$ be a semisimple group algebra. Then*

$$\mathbb{F}G \simeq \mathbb{F}(G/N) \oplus \triangle(G, N),$$

where $\triangle(G, N)$ is an ideal of $\mathbb{F}G$ generated by the set $\{n - 1 : n \in N\}$.

Lemma 2.1 *[27, Lemma 2.7] Let p_1 and p_2 be two primes. Let $\mathbb{F}_{q_1}$ be a field with $q_1 = p_1^{k_1}$ elements and let $\mathbb{F}_{q_2}$ be a field with $q_2 = p_2^{k_2}$ elements, where $k_1, k_2 \geq 1$. Let both the group algebras $\mathbb{F}_{q_1}G$ and $\mathbb{F}_{q_2}G$ be semisimple. Suppose that*

$$\mathbb{F}_{q_1}G \cong \oplus_{i=1}^{t} \mathbf{M}(n_i, \mathbb{F}_{q_1}),\ n_i \geq 1$$

and $M\left(n, \mathbb{F}_{q_2^r}\right)$ is a Wedderburn component of the group algebra $\mathbb{F}_{q_2}G$ for some $r \geq 1$ and any positive integer n, i.e.,

$$\mathbb{F}_{q_2}G \cong \oplus_{i=1}^{s-1} \mathbf{M}(m_i, \mathbb{F}_{q_{2,i}}) \oplus \mathbf{M}(n, \mathbb{F}_{q_2^r}),\ m_i \geq 1,$$

where $\mathbb{F}_{q_{2,i}}$ is a field extension of $\mathbb{F}_{q_2}$. Then $M(n, \mathbb{F}_{q_1})$ must be a Wedderburn component of the group algebra $\mathbb{F}_{q_1}G$, and it appears at least r times in the WD of $\mathbb{F}_{q_1}G$.

Proposition 2.4 *[2, Corollary 3.8] Let $\mathbb{F}G$ be a finite semisimple group algebra, where characteristics of $\mathbb{F}$ is p. If there exists an irreducible representations of degree n over $\mathbb{F}$, then one of the Wedderburn component of $\mathbb{F}G$ is $\mathbf{M}(n, \mathbb{F})$.*

3 Unit group of the group algebra $\mathbb{F}_q GL(2, 7)$

The general linear group of 2×2 matrices over the finite field of order 7 is denoted by G (i.e., $G = GL(2, 7)$). Clearly, $|G| = 2016$. According to Maschke's theorem [18], the group algebra $\mathbb{F}_q G$ is semisimple for $p \neq 2, 3, 7$. Additionally, it can be seen from [12] that G has irreducible representations of the degrees 1, 6, 7 and 8 whenever $| S\mathbb{F}_q(\gamma_g) | = 1$ for all $g \in G$. In order to determine the Wedderburn decomposition (WD) of the group algebra $\mathbb{F}_q G$, we will first look at the unit group of the group algebra $\mathbb{F}_q N$, where $N = PSL(3, 2) \rtimes C_2$, in the following subsection. Later on, we show that one of the factor subgroups of G is isomorphic to N and use the WD of $\mathbb{F}_q N$ to compute that of $\mathbb{F}_q G$.

3.1 WD of the group algebra $\mathbb{F}_q N$, $N = PSL(3, 2) \rtimes C_2$

One can note that the order of N is 336. In this section, we characterize the unit group of the group algebra $\mathbb{F}_q N$ for $p \neq 2, 3, 7$ such that the group algebra $\mathbb{F}_q N$ is semisimple and $q = p^k$. The presentation of N is as follows (we use the notation $[r, s] = r^{-1}s^{-1}rs$):

$$\langle x, y, z \mid x^2(z^{-1}y^{-1})^4 zy^{-1}z^{-1},\ x^{-1}yx(zy^{-1}z^{-1}y^{-1})^2 z^{-1}(y^{-1}z)^2,$$

$$x^{-1}zx(z^{-1}y^{-1})^4 zy^{-1},\ y^2,\ z^3,\ (yz)^7,\ (y^{-1}z^{-1}yz)^4\rangle.$$

Further, using GAP [10], we note that N has 9 conjugacy classes as shown in the table below.

R	e	x^5	x^4	x^3	xzx	x^2y	x^3zx^2y	zx^2y	xzx^2y
S	1	56	56	28	48	21	42	42	42
O	1	6	3	2	7	2	8	4	8

Here, R, S and O denote the representative, size and order of conjugacy classes, respectively. The above discussion clearly indicates that the exponent of N is 168. Let $\mathbb{F}_i$ denote the finite extensions of $\mathbb{F}_q$ and let n_i, r be positive integers.

Theorem 3.1 *The WD of $\mathbb{F}_q N$, where $q = p^k$ and $p \neq 2, 3, 7$, is given as follows:*

(1) *for* $p^k \equiv \{1, 17, 23, 25, 31, 41, 47, 55, 65, 71, 73, 79, 89, 95, 97, 103, 113, 121, 127, 137, 143, 145, 151, 167\} \mod 168$, *we have*

$$\mathbb{F}_q N \simeq \mathbb{F}_q^2 \oplus \mathbf{M}(6, \mathbb{F}_q)^3 \oplus \mathbf{M}(7, \mathbb{F}_q)^2 \oplus \mathbf{M}(8, \mathbb{F}_q)^2;$$

(2) *for* $p^k \equiv \{5, 11, 13, 19, 29, 37, 43, 53, 59, 61, 67, 83, 85, 101, 107, 109, 115, 125, 131, 139, 149, 155, 157, 163\} \mod 168$, *we have*

$$\mathbb{F}_q N \simeq \mathbb{F}_q^2 \oplus \mathbf{M}(6, \mathbb{F}_q) \oplus \mathbf{M}(7, \mathbb{F}_q)^2 \oplus \mathbf{M}(8, \mathbb{F}_q)^2 \oplus \mathbf{M}(6, \mathbb{F}_{q^2}).$$

Proof. The group algebra $\mathbb{F}_q N$ is semisimple. Hence, it follows from the Weddurburn-Artin theorem (see [18]) that $\mathbb{F}_q N \simeq \oplus_{i=1}^r \mathbf{M}(n_i, \mathbb{F}_i)$. The derived subgroup of the group N is $PSL(3, 2)$ and the quotient group is C_2. Along with Proposition 2.2, this gives

$$\mathbb{F}_q N \simeq \mathbb{F}_q^2 \bigoplus_{i=1}^{r-2} \mathbf{M}(n_i, \mathbb{F}_i), \quad n_i \geq 2. \tag{2}$$

The proof is split into the following two cases using Theorem 2.1.
Case 1: $p^k \equiv \{1, 17, 23, 25, 31, 41, 47, 55, 65, 71, 73, 79, 89, 95, 97, 103, 113, 121, 127, 137, 143, 145, 151, 167\} \mod 168$. The cardinality of the cyclotomic $\mathbb{F}_q$-class of γ_g in this case is 1 for every g in N. Using this along with Proposition 2.1 and Theorem 2.2, we rewrite (2) as

$$\mathbb{F}_q N \simeq \mathbb{F}_q^2 \bigoplus_{i=1}^{7} \mathbf{M}(n_i, \mathbb{F}_q),$$

and hence,

$$334 = \sum_{i=1}^{7} n_i^2, \ n_i \geq 2. \tag{3}$$

Since we know that one of the factor subgroups of G is isomorphic to N, Proposition 2.3 confirms that in (3) $n_i \geq 6$ for every i (this holds because G has irreducible representations of the degrees 1, 6, 7 and 8 whenever $| S\mathbb{F}_q(\gamma_g) | = 1$ for all $g \in G$). Consequently, we are remaining with a unique choice given by $(6, 6, 6, 7, 7, 8, 8)$ for n_i's. Hence, the WD is

$$\mathbb{F}_q N \simeq \mathbb{F}_q^2 \oplus \mathbf{M}(6, \mathbb{F}_q)^3 \oplus \mathbf{M}(7, \mathbb{F}_q)^2 \oplus \mathbf{M}(8, \mathbb{F}_q)^2.$$

Case 2: $p^k \equiv \{5, 11, 13, 19, 29, 37, 43, 53, 59, 61, 67, 83, 85, 101, 107, 109, 115, 125, 131, 139, 149, 155, 157, 163\} \mod 168$. In this case, the cyclotomic $\mathbb{F}_q$ classes of γ_g are

$$S\mathbb{F}_q(\gamma_{g_i}) = \{\gamma_{g_i}\} \text{ for } i = 1, \dots, 6, 8, \ S\mathbb{F}_q(\gamma_{g_7}) = \{\gamma_{g_7}, \gamma_{g_9}\}.$$

Applying Proposition 2.1, we derive from (2) that

$$\mathbb{F}_q N \simeq \mathbb{F}_q^2 \bigoplus_{i=1}^{5} \mathbf{M}(n_i, \mathbb{F}_q) \oplus \mathbf{M}(n_6, \mathbb{F}_{q^2}),$$

and hence,

$$334 = \sum_{i=1}^{5} n_i^2 + 2n_6^2, \ n_i \geq 2. \tag{4}$$

According to Lemma 2.1, it is clear that in (4), $n_i \geq 6$ for every i.

There are 3 choices for n_i's given by $(6,6,6,7,7,8)$, $(6,6,6,8,8,7)$ and $(6,7,7,8,8,6)$. To deduce the unique choice, we explicitly take $p = 5$ and $k = 1$. Further, we note that the group N is isomorphic to the group generated by permutations $\langle a, b \rangle$, where $a = (3,8,7,6,5,4)$ and $b = (1,2,6)(3,4,8)$. Next, we consider the map $\Psi : N \to GL(7,5)$ given as follows:

$$a \to \begin{bmatrix} 1 & 0 & 4 & 4 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}, \quad b \to \begin{bmatrix} 4 & 3 & 2 & 2 & 1 & 2 & 1 \\ 0 & 0 & 4 & 4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 4 & 1 & 0 & 0 \\ 2 & 0 & 3 & 4 & 4 & 1 & 4 \\ 0 & 0 & 4 & 4 & 0 & 0 & 1 \end{bmatrix}.$$

Clearly, this map is an irreducible representation of N of degree 7 over $\mathbb{F}_5$, i.e., Ψ is a group homomorphism from N to $GL(7,5)$ and Ψ is irreducible, that is there is no matrix $U \in GL(7,5)$ such that

$$U^{-1}\Psi(g)U = \begin{bmatrix} A(g) & B(g) \\ 0 & C(g) \end{bmatrix} \text{ for all } g \in N,$$

where $A(g)$, $B(g)$ and $C(g)$ are square matrices with entries from $\mathbb{F}_5$ depending on g. Therefore, Proposition 2.4 implies that $\mathbf{M}(7, \mathbb{F}_5)$ must appear in WD of $\mathbb{F}_5 N$. Thus, we are remaining with only two possibilities of n_i's is given by $(6,6,6,7,7,8)$ and $(6,7,7,8,8,6)$. For uniqueness, we again consider a map from $N \to GL(8,5)$ given as follows:

$$a \to \begin{bmatrix} 0 & 0 & 2 & 0 & 1 & 3 & 3 & 2 \\ 0 & 4 & 4 & 1 & 4 & 4 & 4 & 4 \\ 2 & 0 & 1 & 3 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 3 & 2 & 1 & 2 & 3 \\ 0 & 4 & 1 & 4 & 1 & 0 & 0 & 4 \\ 1 & 0 & 1 & 3 & 3 & 3 & 0 & 1 \\ 1 & 4 & 3 & 1 & 1 & 1 & 4 & 2 \\ 4 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad b \to \begin{bmatrix} 3 & 4 & 3 & 2 & 4 & 2 & 1 & 2 \\ 4 & 3 & 2 & 0 & 3 & 0 & 4 & 3 \\ 2 & 4 & 2 & 4 & 2 & 1 & 1 & 4 \\ 2 & 3 & 3 & 3 & 2 & 4 & 0 & 4 \\ 4 & 2 & 3 & 2 & 0 & 2 & 0 & 3 \\ 0 & 0 & 4 & 2 & 2 & 3 & 2 & 3 \\ 2 & 3 & 4 & 4 & 1 & 4 & 1 & 3 \\ 0 & 0 & 2 & 1 & 4 & 4 & 0 & 4 \end{bmatrix}.$$

This means that $\mathbf{M}(8,\mathbb{F}_5)$ must appear in WD of $\mathbb{F}_5 N$. Consequently, the unique choice of n_i's is $(6,7,7,7,8,8,6)$. Thus, the WD of $\mathbb{F}_q N$ is

$$\mathbb{F}_q N \simeq \mathbb{F}_q^2 \oplus \mathbf{M}(6,\mathbb{F}_q) \oplus \mathbf{M}(7,\mathbb{F}_q)^2 \oplus \mathbf{M}(8,\mathbb{F}_q)^2 \oplus \mathbf{M}(6,\mathbb{F}_{q^2}).$$

This completes the proof. □

3.2 WD of the group algebra $\mathbb{F}_q G$

In this section, we determine the WD of the group algebra $\mathbb{F}_q G$, where $G = GL(2,7)$. Using GAP [10], it can be noted that G has 48 conjugacy classes. Let us denote these classes by $[g_i]$, $1 \le i \le 48$, where for each i, g_i is the representative of the i^{th} conjugacy class. Using GAP, we observe that
(i) $g_1, g_3, g_5, g_7, g_9, g_{11}$ are the only elements in their conjugacy classes. Moreover, $|g_1| = 1$, $|g_7| = 2$, $|g_5| = |g_9| = 6$, $|g_3| = |g_{11}| = 3$.
(ii) Each of $g_2, g_4, g_6, g_8, g_{10}, g_{12}$ contains 48 elements in their conjugacy classes. Moreover, $|g_2| = 14$, $|g_4| = |g_{12}| = 21$, $|g_6| = |g_{10}| = 42$, $|g_8| = 7$.
(iii) Each of $g_{13}, g_{14}, \ldots, g_{33}$ have 42 elements in their conjugacy classes. Moreover, $|g_{13}| = 4$, $|g_{14}| = |g_{15}| = 8$, $|g_i| = 48$ for $i = 16,17,18,19,30,31,32,33$, $|g_{20}| = |g_{27}| = 12$, $|g_{21}| = |g_{22}| = |g_{28}| = |g_{29}| = 24$, $|g_i| = 16$ for $i = 23,24,25,26$.
(iv) Each of $g_{34}, g_{35}, \ldots, g_{48}$ contains 56 elements in their conjugacy classes. Moreover, $|g_{34}| = 6$ for $i = 34,35,37,38,39,41,43,44,45,46,48$, $|g_{36}| = 2$, $|g_{40}| = |g_{42}| = |g_{47}| = 3$.

It is clear that the exponent of G is 336. In the following theorem, we determine the WD of the group algebra $\mathbb{F}_q G$ for $p \neq 2,3,7$ and $q = p^k$.

Theorem 3.2 *The WD of $\mathbb{F}_q G$ is as follows:*
(1) *for* $p^k \equiv \{1,55,97,103,145,151,193,199,241,247,289,295\} \mod 336$, *we have*

$$\mathbb{F}_q G \simeq \mathbb{F}_q^6 \oplus \mathbf{M}(6,\mathbb{F}_q)^{21} \oplus \mathbf{M}(7,\mathbb{F}_q)^6 \oplus \mathbf{M}(8,\mathbb{F}_q)^{15};$$

(2) *for* $p^k \equiv \{5,11,29,53,59,83,101,107,125,131,149,155,173,179,197,221,227,251,269,275,293,299,317,323\} \mod 336$, *we have*

$$\mathbb{F}_q G \simeq \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^2 \oplus \mathbf{M}(6,\mathbb{F}_q) \oplus \mathbf{M}(7,\mathbb{F}_q)^2 \oplus \mathbf{M}(8,\mathbb{F}_q)^3 \oplus \mathbf{M}(6,\mathbb{F}_{q^2})^4$$
$$\oplus \mathbf{M}(7,\mathbb{F}_{q^2})^2 \oplus \mathbf{M}(8,\mathbb{F}_{q^2})^6 \oplus \mathbf{M}(6,\mathbb{F}_{q^4})^3;$$

(3) *for* $p^k \equiv \{13,19,37,43,61,67,85,109,115,139,157,163,181,205,211,229,235,253,277,283,307,325,331\} \mod 336$, *we have*

$$\mathbb{F}_q G \simeq \mathbb{F}_q^6 \oplus \mathbf{M}(6,\mathbb{F}_q)^3 \oplus \mathbf{M}(7,\mathbb{F}_q)^6 \oplus \mathbf{M}(8,\mathbb{F}_q)^{15} \oplus \mathbf{M}(6,\mathbb{F}_{q^2})^3 \oplus \mathbf{M}(6,\mathbb{F}_{q^4})^3;$$

(4) *for* $p^k \equiv \{17,23,65,71,113,167,209,215,257,263,305,311\} \mod 336$, *we have*

$$\mathbb{F}_q G \simeq \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^2 \oplus \mathbf{M}(6,\mathbb{F}_q)^7 \oplus \mathbf{M}(7,\mathbb{F}_q)^2 \oplus \mathbf{M}(8,\mathbb{F}_q)^3$$

$$\oplus \mathbf{M}(6, \mathbb{F}_{q^2})^7 \oplus \mathbf{M}(7, \mathbb{F}_{q^2})^2 \oplus \mathbf{M}(8, \mathbb{F}_{q^2})^6;$$

(5) *for* $p^k \equiv \{25, 31, 73, 79, 121, 127, 169, 223, 265, 271, 313, 319\} \mod 336$, *we have*

$$\mathbb{F}_q G \simeq \mathbb{F}_q^6 \oplus \mathbf{M}(6, \mathbb{F}_q)^9 \oplus \mathbf{M}(7, \mathbb{F}_q)^6 \oplus \mathbf{M}(8, \mathbb{F}_q)^{15} \oplus \mathbf{M}(6, \mathbb{F}_{q^2})^6;$$

(6) *for* $p^k \equiv \{41, 47, 89, 95, 137, 143, 185, 191, 233, 239, 281, 335\} \mod 336$, *we have*

$$\mathbb{F}_q G \simeq \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^2 \oplus \mathbf{M}(6, \mathbb{F}_q)^3 \oplus \mathbf{M}7, \mathbb{F}_q)^2 \oplus \mathbf{M}(8, \mathbb{F}_q)^3$$

$$\oplus \mathbf{M}(6, \mathbb{F}_{q^2})^9 \oplus \mathbf{M}(7, \mathbb{F}_{q^2})^2 \oplus \mathbf{M}(8, \mathbb{F}_{q^2})^6.$$

Proof. The group algebra $\mathbb{F}_q G$ is semisimple. Therefore, it follows from the Wedderburn-Artin theorem that $\mathbb{F}_q G \simeq \oplus_{i=1}^r \mathbf{M}(n_i, \mathbb{F}_i)$. Also, it is well known that the derived subgroup of G is $SL(2,7)$, and the factor group is isomorphic to C_6. Along with Proposition 2.2 this gives

$$\mathbb{F}_q G \simeq \mathbb{F}_q^6 \bigoplus_{i=1}^{r-6} \mathbf{M}(n_i, \mathbb{F}_i),\ n_i \geq 2,\ \text{or}\ \mathbb{F}_q G \simeq \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^2 \bigoplus_{i=1}^{r-6} \mathbf{M}(n_i, \mathbb{F}_i),\ n_i \geq 2. \tag{5}$$

The proof is further divided into the following 6 cases, same like with the previous theorem, using the set $T_{G,\mathbb{F}_q}$ of group G.
Case 1: $p^k \equiv \{1, 55, 97, 103, 145, 151, 193, 199, 241, 247, 289, 295\} \mod 336$. In this case, it can be verified that $\mid S\mathbb{F}_q(\gamma_g) \mid = 1$ for all $g \in G$. Using this along with Proposition 2.1, we obtain from (5) that

$$\mathbb{F}_q G \simeq \mathbb{F}_q^6 \bigoplus_{i=1}^{42} \mathbf{M}(n_i, \mathbb{F}_q),$$

and hence,

$$2010 = \sum_{i=1}^{42} n_i^2,\ n_i \geq 2. \tag{6}$$

At the outset of this section, we discussed that there are no irreducible representations of G with degrees 2, 3, 4 and 5 whenever $\mid S\mathcal{K}_q(\gamma_g) \mid = 1$ for all $g \in G$. Also, G has no irreducible representations of degree strictly greater than 8. Therefore, in (6), $8 \geq n_i \geq 6$, $i = 1, 2, \dots 41$.

To uniquely identify the value of n_i's, we take into account the normal subgroup $N = \langle n \rangle$ of G, where $n = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$. We note that the factor group $G/N \cong C_3 \times (PSL(3,2) \rtimes C_2)$. Using part (1) of Theorem 3.1 along with the property of tensor products (see [22]), we derive that

$$\mathbb{F}_q(G/N) \simeq \mathbb{F}_q^6 \oplus \mathbf{M}(6, \mathbb{F}_q)^9 \oplus \mathbf{M}(7, \mathbb{F}_q)^6 \oplus \mathbf{M}(8, \mathbb{F}_q)^6. \tag{7}$$

Substituting (7) in (6) and using Proposition 2.3, we obtain

$$\mathbb{F}_qG \simeq \mathbb{F}_q^6 \oplus \mathbf{M}(6,\mathbb{F}_q)^9 \oplus \mathbf{M}(7,\mathbb{F}_q)^6 \oplus \mathbf{M}(8,\mathbb{F}_q)^6 \bigoplus_{i=1}^{21} \mathbf{M}(n_i,\mathbb{F}_q),$$

and therefore,

$$1008 = \sum_{i=1}^{21} n_i^2,$$

where $8 \geq n_i \geq 6$, $i = 1,2,\ldots,21$. To this end, we are left with only one choice $(6^{12}, 8^9)$. Hence, the WD is

$$\mathbb{F}_qG \simeq \mathbb{F}_q^6 \oplus \mathbf{M}(6,\mathbb{F}_q)^{21} \oplus \mathbf{M}(7,\mathbb{F}_q)^6 \oplus \mathbf{M}(8,\mathbb{F}_q)^{15}.$$

Case 2: $p^k \equiv \{5, 11, 29, 53, 59, 83, 101, 107, 125, 131, 149, 155, 173, 179, 197, 221, 227, 251, 269, 275, 293, 299, 317, 323\} \mod 336$. The cyclotomic $\mathbb{F}_q$ classes of γ_g are

$$S\mathbb{F}_q(\gamma_{g_i}) = \{\gamma_{g_i}\} \text{ for } i = 1,2,7,8,13,36,42,44,$$
$$S\mathbb{F}_q(\gamma_{g_i}) = \{\gamma_{g_i}, \gamma_{g_{i+8}}\} \text{ for } i = 3,4,21,39,$$
$$S\mathbb{F}_q(\gamma_{g_i}) = \{\gamma_{g_i}, \gamma_{g_{i+4}}\} \text{ for } i = 5,6,34,41,$$
$$S\mathbb{F}_q(\gamma_{g_i}) = \{\gamma_{g_i}, \gamma_{g_{i+7}}\} \text{ for } i = 20,40,$$
$$S\mathbb{F}_q(\gamma_{g_{14}}) = \{\gamma_{g_{14}}, \gamma_{g_{15}}\},\ S\mathbb{F}_q(\gamma_{g_{22}}) = \{\gamma_{g_{22}}, \gamma_{g_{28}}\},\ S\mathbb{F}_q(\gamma_{g_{35}}) = \{\gamma_{g_{35}}, \gamma_{g_{37}}\},$$
$$S\mathbb{F}_q(\gamma_{g_{17}}) = \{\gamma_{g_{17}}, \gamma_{g_{19}}, \gamma_{g_{33}}, \gamma_{g_{31}}\},\ S\mathbb{F}_q(\gamma_{g_{23}}) = \{\gamma_{g_{23}}, \gamma_{g_{24}}, \gamma_{g_{25}}, \gamma_{g_{26}}\},$$
$$S\mathbb{F}_q(\gamma_{g_{43}}) = \{\gamma_{g_{43}}, \gamma_{g_{46}}\},\ S\mathbb{F}_q(\gamma_{g_{16}}) = \{\gamma_{g_{16}}, \gamma_{g_{18}}, \gamma_{g_{30}}, \gamma_{g_{32}}\}.$$

Applying Propositions 2.1, 2.2 and Theorem 2.2, we derive from (5) that

$$\mathbb{F}_qG \simeq \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^2 \bigoplus_{i=1}^{6} \mathbf{M}(n_i,\mathbb{F}_q) \bigoplus_{i=7}^{18} \mathbf{M}(n_i,\mathbb{F}_{q^2}) \bigoplus_{i=19}^{21} \mathbf{M}(n_i,\mathbb{F}_{q^4}),$$

and therefore,

$$2010 = \sum_{i=1}^{6} n_i^2 + 2\sum_{i=7}^{18} n_i^2 + 4\sum_{i=19}^{21} n_i^2,\ n_i \geq 2. \tag{8}$$

We consider the same normal subgroup as considered in Case 1 and observe that the WD of $\mathbb{F}_q(G/N)$ in this case is

$$\begin{aligned}\mathbb{F}_q(G/N) \simeq\ & \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^2 \oplus \mathbf{M}(6,\mathbb{F}_q) \oplus \mathbf{M}(6,\mathbb{F}_{q^2})^4 \oplus \mathbf{M}(7,\mathbb{F}_q)^2 \\ & \oplus \mathbf{M}(7,\mathbb{F}_{q^2})^2 \oplus \mathbf{M}(8,\mathbb{F}_q)^2 \oplus \mathbf{M}(8,\mathbb{F}_{q^2})^2.\end{aligned} \tag{9}$$

Using (8), (9) and Proposition 2.3, we further derive that

$$\begin{aligned}\mathbb{F}_qG \simeq\ & \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^2 \oplus \mathbf{M}(6,\mathbb{F}_q) \oplus \mathbf{M}(6,\mathbb{F}_{q^2})^4 \oplus \mathbf{M}(7,\mathbb{F}_q)^2 \oplus \mathbf{M}(7,\mathbb{F}_{q^2})^2 \\ & \oplus \mathbf{M}(8,\mathbb{F}_q)^2 \oplus \mathbf{M}(8,\mathbb{F}_{q^2})^2 \oplus \mathbf{M}(n_1,\mathbb{F}_q) \bigoplus_{i=2}^{5} \mathbf{M}(n_i,\mathbb{F}_{q^2}) \bigoplus_{i=6}^{8} \mathbf{M}(n_i,\mathbb{F}_{q^4})\end{aligned}$$

with

$$1008 = n_1^2 + \sum_{i=2}^{5} 2n_i^2 + \sum_{i=6}^{8} 4n_i^2, \ n_i \geq 2. \tag{10}$$

In accordance with Lemma 2.1 and Case 1 in (10), we must have $6 \leq n_i \leq 8$ for all $i = 1, 2, \dots, 8$. Thus, we are remaining with 3 choices of n_i's given by $(8, 6^5, 8^2)$, $(8, 6^2, 8^2, 6^2, 8)$ and $(8^5, 6^3)$. We explicitly take $p = 5$ and $k = 1$. With these parameters, $\mathbf{M}(6, \mathbb{F}_{5^4})$ contains a subgroup isomorphic to $GL(2, 7)$, whereas $\mathbf{M}(8, \mathbb{F}_{5^4})$ does not contain any such subgroup. Consequently, the required choice of n_i's is $(8^5, 6^3)$. Hence, we get

$$\mathbb{F}_qG \simeq \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^2 \oplus \mathbf{M}(6, \mathbb{F}_q) \oplus \mathbf{M}(7, \mathbb{F}_q)^2 \oplus \mathbf{M}(8, \mathbb{F}_q)^3 \oplus \mathbf{M}(6, \mathbb{F}_{q^2})^4$$
$$\oplus \mathbf{M}(7, \mathbb{F}_{q^2})^2 \oplus \mathbf{M}(8, \mathbb{F}_{q^2})^6 \oplus \mathbf{M}(6, \mathbb{F}_{q^4})^3.$$

Case 3: $p^k \equiv \{13, 19, 37, 43, 61, 67, 85, 109, 115, 139, 157, 163, 181, 205, 211, 229, 235, 253, 277, 283, 307, 325, 331\} \mod 336$. Similar to the previous case, we note that 30 cyclotomic classes of γ_g, $g \in G$ have single elements in their classes, 3 cyclotomic classes have two elements in their classes and the rest 3 have four elements. Applying Proposition 2.1 and Theorem 2.2, we derive from (5)

$$\mathbb{F}_qG \simeq \mathbb{F}_q^6 \bigoplus_{i=1}^{24} \mathbf{M}(n_i, \mathbb{F}_q) \bigoplus_{i=25}^{27} \mathbf{M}(n_i, \mathbb{F}_{q^2}) \bigoplus_{i=28}^{30} \mathbf{M}(n_i, \mathbb{F}_{q^4}),$$

and hence,

$$2010 = \sum_{i=1}^{24} n_i^2 + 2\sum_{i=25}^{27} n_i^2 + 4\sum_{i=28}^{30} n_i^2, \ n_i \geq 2. \tag{11}$$

In this case, we have

$$\mathbb{F}_q(G/N) \simeq \mathbb{F}_q^6 \oplus \mathbf{M}(6, \mathbb{F}_q)^3 \oplus \mathbf{M}(6, \mathbb{F}_{q^2})^3 \oplus \mathbf{M}(7, \mathbb{F}_q)^6 \oplus \mathbf{M}(8, \mathbb{F}_q)^6. \tag{12}$$

Using (12), (11) and Proposition 2.3, we further derive

$$\begin{aligned}\mathbb{F}_qG \simeq\ & \mathbb{F}_q^6 \oplus \mathbf{M}(6, \mathbb{F}_q)^3 \oplus \mathbf{M}(6, \mathbb{F}_{q^2})^3 \oplus \mathbf{M}(7, \mathbb{F}_q)^6 \oplus \mathbf{M}(8, \mathbb{F}_q)^6 \\ & \bigoplus_{i=1}^{9} \mathbf{M}(n_i, \mathbb{F}_q) \bigoplus_{i=10}^{12} \mathbf{M}(n_i, \mathbb{F}_{q^4})\end{aligned} \tag{13}$$

with

$$1008 = \sum_{i=1}^{9} n_i^2 + 4\sum_{i=10}^{12} n_i^2, \ n_i \geq 2. \tag{14}$$

In accordance with Lemma 2.1 and Case 1 in (14), we must have $6 \leq n_i \leq 8$, $i = 1, 2, \dots 12$. This leaves us with the following three choices for the values of n_i's: $(6^8, 8, 6, 8^2)$, $(6^4, 8^5, 6^2, 8)$ and $(8^9, 6^3)$. Further, as in Case 2 of Theorem 3.1, one can show that there are more than 5 irreducible representations of G of degree 8 over $\mathbb{F}_{13}$. This shows that the final choice of n_i's is $(8^9, 6^3)$. Hence, (13) implies that the WD is

$$\mathbb{F}_q G \simeq \mathbb{F}_q^6 \oplus \mathbf{M}(6, \mathbb{F}_q)^3 \oplus \mathbf{M}(7, \mathbb{F}_q)^6 \oplus \mathbf{M}(8, \mathbb{F}_q)^{15} \oplus \mathbf{M}(6, \mathbb{F}_{q^4})^3.$$

Case 4: $p^k \equiv \{17, 23, 65, 71, 113, 167, 209, 215, 257, 263, 305, 311\} \mod 336$. This case can be done similarly to Case 2 (or Case 3). The WD in this case is

$$\mathbb{F}_q G \simeq \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^2 \oplus \mathbf{M}(6, \mathbb{F}_q)^7 \oplus \mathbf{M}(7, \mathbb{F}_q)^2 \oplus \mathbf{M}(8, \mathbb{F}_q)^3 \oplus \mathbf{M}(6, \mathbb{F}_{q^2})^7$$
$$\oplus \mathbf{M}(7, \mathbb{F}_{q^2})^2 \oplus \mathbf{M}(8, \mathbb{F}_{q^2})^6.$$

Case 5: $p^k \equiv \{25, 31, 73, 79, 121, 127, 169, 223, 265, 271, 313, 319\} \mod 336$. Applying similar to Case 3 reasoning, we get

$$\mathbb{F}_q G \simeq \mathbb{F}_q^6 \oplus \mathbf{M}(6, \mathbb{F}_q)^9 \oplus \mathbf{M}(7, \mathbb{F}_q)^6 \oplus \mathbf{M}(8, \mathbb{F}_q)^{15} \oplus \mathbf{M}(6, \mathbb{F}_{q^2})^6.$$

Case 6: $p^k \equiv \{41, 47, 89, 95, 137, 143, 185, 191, 233, 239, 281, 335\} \mod 336$. The WD in this case is

$$\mathbb{F}_q G \simeq \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^2 \oplus \mathbf{M}(6, \mathbb{F}_q)^3 \oplus \mathbf{M}(7, \mathbb{F}_q)^2 \oplus \mathbf{M}(8, \mathbb{F}_q)^3 \oplus \mathbf{M}(6, \mathbb{F}_{q^2})^9$$
$$\oplus \mathbf{M}(7, \mathbb{F}_{q^2})^2 \oplus \mathbf{M}(8, \mathbb{F}_{q^2})^6,$$

which can be shown analogously to the previous cases. This completes the proof. □

It is straightforward to compute the unit group of $\mathbb{F}_q G$ from Theorem 3.2.

4 Conclusion

In this paper, we computed the unit group for the semisimple group algebra of the group $GL(2,7)$. For this, we calculated the Wedderburn decomposition of the group algebra by using the results from the classical theory of group algebras. It is clear that as the group size increases, it becomes difficult to characterize the Wedderburn decomposition due to the large range of potential Wedderburn components. The study motivates the determination of the unit group of groups algebras of the general linear groups of higher order by discovering new techniques to reduce the large range of potential Wedderburn components.

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