Armenian Journal of Mathematics
Volume 16, Number 9, 2024, 1–18
https://doi.org/10.52737/18291163-2026.16.09-1-18

Differential Subordination and Coefficient Functionals of Univalent Functions Related to $\cos z$

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In the memory of Shri Inder Lal Sharma

Abstract. Differential subordination in the complex plane is the generalization of a differential inequality on the real line. In this paper, we consider two subclasses of univalent functions associated with the trigonometric function $\cos z$. Using some properties of the hypergeometric functions, we determine the sharp estimate on the parameter β such that the analytic function $p(z)$ satisfying $p(0) = 1$, is subordinate to $\cos z$ when the differential expression $p(z) + \beta z(dp(z)/dz)$ is subordinate to the Janowski function. We compute sharp bounds on coefficient functional Hermitian–Toeplitz determinants of the third and the fourth order with an invariance property for such functions. In addition, we estimate bound on Hankel determinants of the second and the third order.

Key Words: Differential Subordination, Univalent Functions, Starlike Functions, Convex Function, $\cos z$, Hermitian–Toeplitz Determinant, Hankel Determinant
Mathematics Subject Classification 2020: 30C45, 30C50

Introduction

Differential subordination is a mathematical tool that helps analyzing and comparing the behavior of analytic functions in the complex plane. It is particularly useful in the study of univalent functions and their properties. The estimates on coefficient functionals have many important consequences in univalent function theory, including the Koebe distortion theorem which describes how conformal maps distort shapes. It was named after Paul

Koebe, a German mathematician who first proved mentioned theorem in 1907.

Further, in 1916, Ludwig Bieberbach proposed the Bieberbach conjecture and other coefficient inequalities for a univalent function, in particular, that the radius of univalence of f is at least $1/4$. That is, for any function $w \in \mathbb{C}$ with $|w| < 1/4$, there exists a unique z in the open unit disk such that $f(z) = w$. This study inspired many researchers to interrogate the coefficient functionals like the Hankel determinant and Hermitian–Toeplitz determinant. These determinants play important role in several branches of mathematics, especially, in operator theory, matrix measure, matrix polynomial, signal processing, time series analysis, integral equations, as well as univalent function theory (see, for example, [7, 8]).

We denote by $\mathcal{A}$ the class of all analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ containing univalent functions. We denote by $\mathcal{P}$ the class of all analytic functions p in $\mathbb{D}$ satisfying $p(0) = 1$ and $\operatorname{Re} p(z) > 0$. The analytic function h is said to be convex or starlike if $h(\mathbb{D})$ is a convex or starlike domain, respectively. In view of Alexander's theorem, the function h is convex if and only if $zh'(z)$ is starlike. Denote by $\mathcal{S}^*$ and $\mathcal{K}$ the subclasses of $\mathcal{S}$ containing starlike and convex functions, respectively. These classes have a number of interesting properties and are useful in the study of various problems in complex analysis and geometry. For example, starlike functions can be used to study problems involving conformal mappings of simply connected domains, while convex functions can be used to study problems involving minimal surfaces and the isoperimetric inequality (see [7]).

Let g_1 and g_2 be analytic functions defined in $\mathbb{D}$. The function g_1 is said to be subordinate to g_2, denoted by $g_1 \prec g_2$, if there exists a Schwarz function w such that $g_1 = g_2 \circ w$. In particular, if the function g_2 is in the subclass $\mathcal{S}$, then $g_1 \prec g_2$ if and only if $g_1(0) = g_2(0)$ and $g_1(\mathbb{D}) \subseteq g_2(\mathbb{D})$ (see [7]).

In 1971, Janowski [9] considered the subclass $\mathcal{S}^*[A, B]$ which consists of functions $f \in \mathcal{A}$ satisfying the relation

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}$$

where $-1 \leq B < A \leq 1$. In particular, if $A = 1$ and $B = -1$, the class $\mathcal{S}^*[A, B]$ reduces to the class $\mathcal{S}^*$.

In 2020, Tang *et al.* [34] introduced two subclasses $\mathcal{S}^*_{\cos}$ and $\mathcal{K}_{\cos}$ of univalent functions which consist of starlike and convex functions associated with the trigonometric function $\cos z$. In terms of subordination, these subclasses are defined as

$$\mathcal{S}^*_{\cos} := \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \cos z \right\} \text{ and } \mathcal{K}_{\cos} = \left\{ f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} \prec \cos z \right\}$$

for all $z \in \mathbb{D}$. The function $\cos z$ maps $\mathbb{D}$ into domain $\{w \in \mathbb{C} : |\cos^{-1} w| < 1\}$. The cosine function is a periodic and entire function which is eventually used in the study of sound and light waves. Using the concept of differential subordination, Bano and Raza [4] studied the class $\mathcal{S}^*_{\cos}$ and its geometric properties like structural formula, radii problems, inclusion relations and sufficient condition for certain starlikeness. It was noted that $f \in \mathcal{S}^*_{\cos}$ if there exists an analytic function $h(z) \prec h_0(z) = \cos z$ such that

$$f(z) = z \exp\left(\int_0^z \frac{h(u) - 1}{u} du \right),$$

which is the structural formula for subclass $\mathcal{S}^*_{\cos}$. Taking $h = h_0$, we have

$$f(z) = z \exp\left(\int_0^z \frac{\cos u - 1}{u} du \right).$$

The function f plays a role of an extremal function for many geometric problems of the class $\mathcal{S}^*_{\cos}$ (see, for example, [4]).

For natural numbers q and n, Hermitian–Toeplitz determinant of the q^{th} order allied with the coefficients a_n in the series expansion of the functions $f \in \mathcal{A}$ is given by

$$T_q(n) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ \overline{a}_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ \overline{a}_{n+q-1} & \overline{a}_{n+q-2} & \cdots & a_n \end{vmatrix}.$$

Particularly,

$$T_3(1) := 2\,\mathrm{Re}(a_2^2 \overline{a_3}) - 2|a_2|^2 - |a_3|^2 + 1, \tag{1}$$
$$T_4(1) := 1 - 2\mathrm{Re}(a_2^3 \bar{a}_4) + 4\mathrm{Re}(a_2^2 \bar{a}_3) - 2\mathrm{Re}(a_2 \bar{a}_3{}^2 a_4) + 4\mathrm{Re}(a_2 a_3 \bar{a}_4) + |a_2|^4 - |a_2|^2 + |a_3|^4 - 2|a_3|^2 + |a_2|^2|a_4|^2 - 2|a_2|^2|a_3|^2 - |a_4|^2.$$

In a similar way, Hankel determinant of the n^{th} order of the functions $f \in \mathcal{A}$ is given by

$$H_q(n) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)}. \end{vmatrix}.$$

Particularly,

$$H_2(3) := a_3 a_5 - a_4^2, \qquad H_3(1) := 2a_2 a_3 a_4 - a_2^2 a_5 - a_3^3 + a_3 a_5 - a_4^2.$$

Nunokawa *et al.* [23] established the first order differential subordination that states $p(z) \prec 1 + z$ if $1 + zp'(z) \prec 1 + z$. Using the technique due to Ruscheweyh [30], Ali *et al.* [2] established subordination relation between function $p \in \mathcal{P}$ and the Janowski function $(1 + Az)/(1 + Bz)$ where $A, B \in [-1, 1]$. These non-sharp results yield sufficient conditions for the functions to be in the class $\mathcal{S}^*[A, B]$. In [17], the sharp estimates on β were computed such that the function p is subordinate to certain functions with positive real part whenever $1+\beta zp'(z)/p^j(z)$ $(j = 0, 2)$ is subordinate to the Janowski function. Bohra *et al.* [5] investigated differential subordination inclusions for certain functions with positive real parts using properties of hypergeometric functions. Srivastava and Kareem [33] gave some applications of the first order differential subordinations for holomorphic functions in complex normed spaces.

The bounds on $T_3(1)$ for the classes of starlike and convex functions were determined in [6]. Further, the sharp bounds on $T_3(1)$ for close-to-star functions were computed in [10]. Rai *et al.* [26] computed bounds on $T_3(1)$ for the starlike functions associated with tan hyperbolic functions. Lecko *et al.* [20] computed sharp estimates on $T_4(1)$ for convex functions. For more details, we refer to [16, 18, 19, 24, 27, 32].

Hayman [8] and Pommerenke [25] computed bounds on Hankel determinants for certain univalent functions. Sim *et al.* [31] obtained the sharp bound on the second Hankel determinant for the classes of strongly starlike and strongly convex functions of order β. In 2018, using various inequalities related to function $p \in \mathcal{P}$, Zaprawa [35] determined a bound on $H_2(3)$ for the starlike and convex functions under additional condition. Babalola [3] was the first to discuss the problems of estimating $H_3(1)$ for starlike and convex functions, which were finally solved in [11] and [12]. In [28], the sharp bound on $H_3(1)$ for starlike functions of order 1/2 was obtained.

In this paper, we determine sharp estimate on the parameter β such that the differential subordination relation $p(z) \prec \cos z$ holds whenever the differential subordination $p(z) + \beta zp'(z) \prec (1 + Az)/(1 + Bz)$ holds, $-1 \leq B < A \leq 1$. Further, we obtain sharp estimates on coefficient functional Hermitian–Toeplitz determinants $T_3(1)$ and $T_4(1)$ with an invariance property for functions f belonging to classes $\mathcal{S}^*_{\cos}$ and $\mathcal{K}_{\cos}$, respectively. We also determine bounds on $H_2(3)$ and $H_3(1)$ for such functions.

1 Differential subordination

In this section, we find sharp estimates on the parameter β such that the analytic function $p(z)$ is subordinate to $\cos z$ whenever $p(z) + \beta z(dp(z)/dz)$ is subordinate to the Janowski function $(1 + Az)/(1 + Bz)$ where $-1 \leq B < A \leq 1$.

Theorem 1 *Assume*

$$\chi(\beta, A, B) := -\frac{A-B}{\beta+1}\sum_{j=0}^{\infty}\frac{\Gamma(j)}{(j-1)!(1+\beta+j\beta)}B^j + \frac{\beta}{\beta+1} + \frac{1}{\beta+1}$$

and

$$\xi(\beta, A, B) := \frac{(A-B)}{\beta+1}\sum_{j=0}^{\infty}\frac{\Gamma(j)}{(j-1)!(1+\beta+j\beta)}(-B)^j - \frac{\beta}{\beta+1} - \frac{1}{\beta+1}$$

where $-1 \leq B < A \leq 1$. *Let function* $p \in \mathcal{P}$ *satisfies*

$$p(z) + \beta z\frac{dp(z)}{dz} \prec \frac{1+Az}{1+Bz}.$$

If $\beta \geq \max\{\beta_1, \beta_2\}$, *then* $p(z) \prec \cos z$, *where* β_1 *and* β_2 *are positive roots of equations*

$$\chi(\beta, A, B) = \cos 1 \qquad \text{and} \qquad \xi(\beta, A, B) = \cos 1, \tag{2}$$

respectively. The bound on β *is sharp.*

In the proof of this result, we use some properties of hypergeometric functions and results due to Küstner [14] and Miller and Mocanu [22] presented below.

Let $(a)_k$ denote the Pochhammer symbol given by $(a)_k = \Gamma(a+k)/\Gamma(a) = a(a+1)\cdots(a+k-1)$ and $(a)_0 = 1$. For $|z| < 1$ and parameters $a, b \in \mathbb{C}$, $c \notin \{0 \cup \mathbb{Z}_-\}$, the hypergeometric function ${}_2F_1(a, b; c; z)$ is defined by the convergent power series

$$F(a, b; c; z) = {}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}z^k. \tag{3}$$

The function $F(a, b; c; z)$ is analytic in $\mathbb{C}$ and is one of the solutions to the differential equation $z(1-z)y'' + [c-(a+b+1)z]y' - aby = 0$ at $z = 0$. The derivative of the function $F(a, b; c; z)$ satisfies the relation

$$\frac{\partial F(a, b; c; z)}{\partial z} = \frac{ab}{c}F(a+1, b+1; c+1; z).$$

For an analytic function f, its order of starlikeness with respect to zero is defined as follows:

$$\sigma(f) := \inf_{z\in\mathbb{D}} \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \in [-\infty, 1].$$

Theorem 2 *[14, Theorem 1(a)] Let a, b and c be non-zero real numbers such that $0 < a \leq b \leq c$. Then*

$$1 - \frac{ab}{b+c} \leq \sigma(zF(a,b;c;z)) \leq 1 - \frac{ab}{2c}.$$

Lemma 1 *[22, Theorem 3.4h, p.132] Let $q : \mathbb{D} \to \mathbb{C}$ be an analytic function and let ψ and v be analytic functions in a domain $U \supseteq q(\mathbb{D})$ with $\psi(w) \neq 0$ whenever $w \in q(\mathbb{D})$. Set*

$$Q(z) := zq'(z)\psi(q(z)) \quad \textit{and} \quad h(z) := v(q(z)) + Q(z), z \in \mathbb{D}.$$

Suppose that

(i) either $h(z)$ is convex or $Q(z)$ is starlike univalent in $\mathbb{D}$;

(ii) $\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) > 0$, $z \in \mathbb{D}$.

If p is analytic in $\mathbb{D}$ with $p(0) = q(0)$, $p(\mathbb{D}) \subset U$, and

$$v(p(z)) + zp'(z)\psi(p(z)) \prec v(q(z)) + zq'(z)\psi(q(z)),$$

then $p \prec q$. Here, q is the best dominant.

Proof of Theorem 1 Let

$$q_\beta(z) := \frac{A-B}{\beta+1} z \left(F(1, 1+\beta^{-1}; 2+\beta^{-1}; -Bz)\right) + \frac{\beta}{\beta+1} + \frac{1}{\beta+1}$$

be the analytic solution to the differential equation

$$\beta z \frac{dq}{dz} + q = \frac{1+Az}{1+Bz}, \qquad z \in \mathbb{D}.$$

For $w \in \mathbb{C}$, define $v(w) := w$ and $\psi(w) := \beta$. Then

$$\begin{aligned} Q(z) &= zq'_\beta(z)\psi(q_\beta(z)) \\ &= \beta z q'_\beta(z) \\ &= \beta z \Bigg[\frac{A-B}{\beta+1}\left(F(1, 1+\beta^{-1}; 2+\beta^{-1}; -Bz)\right) \\ &\qquad\qquad + \frac{A-B}{2\beta+1} z \left(F(2, 2+\beta^{-1}; 3+\beta^{-1}; -Bz)\right)\Bigg]. \end{aligned}$$

From the hypergeometric functions $F(a, b; c; z)$ defined in (3) and the function $F(2, 2+\beta^{-1}; 3+\beta^{-1}; -Bz)$, we get $a = 2, b = 2+\beta^{-1}$ and $c = 3+\beta^{-1}$, and hence, $0 < a \leq b \leq c$. Since $\beta > 0$, we have

$$\sigma\left(zF(2, 2+\beta^{-1}; 3+\beta^{-1}; -Bz)\right) \geq 1 - \frac{2+4\beta}{2+5\beta} = \frac{\beta}{2+5\beta} > 0.$$

Therefore, the hypergeometric function $zF(2, 2+\beta^{-1}; 3+\beta^{-1}; -Bz)$ is starlike, which ensures the starlikeness of the function Q. Since $\beta > 0$ and Q is starlike, the function $h(z) = v(q_\beta(z)) + Q(z) = q_\beta(z) + Q(z)$ satisfies

$$\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) = \operatorname{Re}\left(\frac{1}{\beta} + \frac{zQ'(z)}{Q(z)}\right) = \frac{1}{\beta} + \operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right) > 0$$

for all $z \in \mathbb{D}$. Due to Lemma 1, if $p(z) + \beta zp'(z) \prec q_\beta(z) + \beta zq'_\beta(z)$, then $p \prec q_\beta$. Note that the subordination is transitive. It is enough to show that $q_\beta(z) \prec \cos z$ for the required subordination $p(z) \prec \cos z$ to hold. Since

$$q_\beta(-1) = -\frac{A-B}{\beta+1}\left(F(1, 1+\beta^{-1}; 2+\beta^{-1}; B)\right) + \frac{\beta}{\beta+1} + \frac{1}{\beta+1}$$

and

$$q_\beta(1) = \frac{A-B}{\beta+1}\left(F(1, 1+\beta^{-1}; 2+\beta^{-1}; -B)\right) + \frac{\beta}{\beta+1} + \frac{1}{\beta+1},$$

the subordination $q_\beta \prec \cos z$ holds if

$$\cos(-1) \leq q_\beta(-1) \leq q_\beta(1) \leq \cos 1.$$

The above inequalities reduce to

$$-\frac{A-B}{\beta+1}\sum_{j=0}^{\infty}\frac{\Gamma(j)}{(j-1)!(1+\beta+j\beta)}B^j + \frac{\beta}{\beta+1} + \frac{1}{\beta+1} - \cos(-1) \geq 0$$

and

$$\cos 1 - \frac{(A-B)}{\beta+1}\sum_{j=0}^{\infty}\frac{\Gamma(j)}{(j-1)!(1+\beta+j\beta)}(-B)^j - \frac{\beta}{\beta+1} - \frac{1}{\beta+1} \geq 0.$$

Therefore, $q_\beta \prec \cos z$ if $\beta \geq \max\{\beta_1, \beta_2\}$, where β_1 and β_2 are positive roots of the equations given in (2). □

The sufficient condition for cosine starlikeness is given below.

Corollary 1 *Let $A, B \in [-1, 1]$ and $f \in \mathcal{A}$ be such that*

$$\frac{z}{f(z)}\left((1+\beta)f'(z) + \beta z\left(f''(z) - \frac{f'(z)^2}{f(z)}\right)\right) \prec \frac{1+Az}{1+Bz}.$$

Then $f \in S^*_{\cos}$ if $\beta \geq \max\{\beta_1, \beta_2\}$, where β_1 and β_2 are positive roots of the equations given in (2).

2 Hermitian–Toeplitz determinants

In this section, we provide the sharp bounds on $T_3(1)$ and $T_4(1)$ for the subclasses $\mathcal{S}^*_{\cos}$ and $\mathcal{K}_{\cos}$ with an invariance property. We use the following result due to Libra in the demonstration of proof.

Lemma 2 [21, Lemma 3, p. 254] *Let $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$ be in the class $\mathcal{P}$. Then $2p_2 = p_1^2 + (4 - p_1^2)\xi$ for some $\xi \in \overline{\mathbb{D}}$.*

Theorem 3 *Let function $f \in \mathcal{A}$ be in the class $\mathcal{S}^*_{\cos}$. Then*

$$\frac{15}{16} \leq T_3(1) \leq 1 \quad \textit{and} \quad \frac{225}{256} \leq T_4(1) \leq 1.$$

Proof. For $f \in \mathcal{S}^*_{\cos}$, we have

$$zf'(z) = f(z)\cos(w(z)) \qquad \text{for all } z \in \mathbb{D},$$

where w is the Schwarz function. Since $p(z) = (1 + w(z))/(1 - w(z)) \in \mathcal{P}$, we get

$$\frac{zf'(z)}{f(z)} = \cos\left[\frac{p(z) - 1}{p(z) + 1}\right],$$

which gives

$$\begin{aligned}\frac{zf'(z)}{f(z)} = {} & 1 + a_2 z + (2a_3 - a_2^2)z^2 + (a_2^3 - 3a_2a_3 + 3a_4)z^3 \\ & + (-a_2^4 + 4a_2^2a_3 - 4a_2a_4 - 2a_3^2 + 4a_5)z^4 \\ & + (a_2^5 - 5a_2^3a_3 + 5a_2^2a_4 + 5a_2a_3^2 - 5a_2a_5 - 5a_3a_4 + 5a_6)z^5 + \cdots\end{aligned}$$

and

$$\begin{aligned}\cos\left[\frac{p(z) - 1}{p(z) + 1}\right] = {} & 1 - \frac{p_1^2}{8}z^2 + \frac{1}{8}p_1(p_1^2 - 2p_2)z^3 \\ & + \frac{1}{384}(-35p_1^4 + 144p_1^2p_2 - 96p_1p_3 - 48p_2^2)z^4 \\ & + \frac{1}{192}(11p_1^5 - 70p_1^3p_2 + 72p_1^2p_3 + 72p_1p_2^2 - 48p_1p_4 \\ & - 48p_2p_3)z^5 + \cdots \end{aligned} \tag{4}$$

Equating the coefficients of the same powers of z, we obtain

$$a_2 = 0, \tag{5}$$

$$a_3 = -\frac{p_1^2}{16}, \tag{6}$$

$$a_4 = \frac{1}{24}p_1(p_1^2 - 2p_2), \tag{7}$$

$$a_5 = \frac{1}{96}\left(-2p_1^4 + 9p_1^2p_2 - 3p_2^2 - 6p_1p_3\right), \tag{8}$$

$$a_6 = \frac{1}{1920}(17p_1^5 - 130p_1^3p_2 + 144p_1^2p_3 - 96p_2p_3 + 48p_1(3p_2^2 - 2p_4)).$$

Then

$$T_3(1) = 2\mathrm{Re}(a_2^2\bar{a}_3) - 2|a_2|^2 - |a_3|^2 + 1 = 1 - \frac{|p_1|^4}{256}.$$

It is easy to verify that the subclasses $\mathcal{P}$ and $\mathcal{S}^*_{\cos}$ are rotationally invariant. Thus, we have $0 \leq p_1 \leq 2$. Next, set $p^2 =: x \in [0,4]$ such that $T_3(1) = 1 - x^2/256$ for all $x \in [0,4]$. Thus, we get the minimum and the maximum values of $T_3(1)$ as desired. The lower bound is sharp for the function

$$f_1(z) = z\exp\left(\int_0^z \frac{\cos t - 1}{t}dt\right) = z - \frac{1}{4}z^3 + \frac{1}{24}z^5 + \cdots, \tag{9}$$

and the upper bound on $T_3(1)$ is sharp for the function

$$f_2(z) = z\exp\left(\int_0^z \frac{\cos t^2 - 1}{t}dt\right) = z + \frac{1}{8}z^5 + \cdots.$$

Substituting the obtained values of a_2, a_3 and a_4 from (5), (6) and (7) in expression (1) of the fourth Hermitian–Toeplitz determinant, we have

$$T_4(1) = 1 + \frac{|p_1|^8}{16^4} - \frac{|p_1|^4}{128} - \frac{1}{24^2}|p_1|^2|p_1^2 - 2p_2|^2. \tag{10}$$

Using Lemma 2, we get

$$\begin{aligned} |p_1^2 - 2p_2|^2 &= p_1^4 + |p_1^2 + (4-p_1^2)\xi|^2 - 2p_1^2(p_1^2 + (4-p_1^2)\mathrm{Re}(\bar{\xi}) \\ &= (4-p_1^2)^2|\xi|^2 \end{aligned} \tag{11}$$

for some $\xi \in \overline{\mathbb{D}}$. From expressions (11) and (10), we have

$$\begin{aligned} T_4(1) &= 1 + \frac{|p_1^8|}{16^4} - \frac{|p_1|^4}{128} - \frac{1}{576}|p_1|^2(4-p_1^2)^2|\xi|^2 \\ &= 1 + \frac{1}{64}\left[\frac{1}{1024}p_1^8 - \frac{1}{2}p_1^4 - \frac{1}{9}p_1^2(4-p_1^2)^2|\xi|^2\right] \end{aligned}$$

Consider $p^2 =: x \in [0,4]$ and $|\xi| =: y \in [0,1]$ such that

$$T_4(1) = 1 + \frac{1}{64}\left[\frac{x^4}{1024} - \frac{x^2}{2} - \frac{1}{9}x(4-x)^2y^2\right] = G(x,y).$$

Using second derivative test, the maximum value of $G(x,y)$ is 1 and the minimum value of $G(x,y)$ is $225/256$ in the region $[0,4] \times [0,1]$. Thus, we get the required estimates on $T_4(1)$. The lower bound on $T_4(1)$ is sharp for the function f_1 defined by (9) and the upper bound on $T_4(1)$ is sharp for the function

$$f_3(z) = z\exp\left(\int_0^z \frac{\cos t^3 - 1}{t}dt\right) = z - \frac{1}{12}z^7 + \cdots.$$

□

Since the function $f(z) = z + a_2z^2 + a_3z^3 + \cdots = \omega$ is analytic, one-to-one in $\mathbb{D}$ and its inverse is $f^{-1}(\omega) = \omega + A_2\omega^2 + A_3\omega^3 + A_4\omega^4 + \cdots$ in some neighbourhood of origin, we have $f(f^{-1}(\omega)) = \omega = f(z)$. Thus, the initial inverse coefficients are given by

$$A_2 = -a_2, \quad A_3 = 2a_2^2 - a_3 \quad \text{and} \quad A_4 = -5a_2^3 + 5a_2a_3 - a_4, \tag{12}$$

respectively. For initial details, see [1, 13].

Since $f \in \mathcal{S}^*_{\cos}$, in view of (5), we get $A_2 = 0$, $A_3 = -a_3$ and $A_4 = -a_4$. Therefore, for inverse coefficients, the third and the fourth order Hermitian–Toeplitz determinants become

$$T_3(1)(f^{-1}) = 1 - |A_3|^2 \quad \text{and} \quad T_4(1)(f^{-1}) = 1 + |A_3|^4 - 2|A_3|^2 - |A_4|^2,$$

respectively. Note also that $T_3(1)(f^{-1}) = T_3(1)$ and $T_4(1)(f^{-1}) = T_4(1)$. Therefore, for the functions $f \in \mathcal{S}^*_{\cos}$, one has an invariance property between Hermitian–Toeplitz determinants of the third and the fourth order involving initial coefficients and inverse coefficients, respectively.

Corollary 2 *Let $f \in \mathcal{S}^*_{\cos}$. Then*

$$\frac{15}{16} \leq T_3(1), T_3(1)(f^{-1}) \leq 1 \quad \text{and} \quad \frac{225}{256} \leq T_4(1), T_4(1)(f^{-1}) \leq 1.$$

Theorem 4 *Let $f \in \mathcal{K}_{\cos}$. Then*

$$\frac{143}{144} \leq T_3(1) \leq 1 \quad \text{and} \quad \frac{20449}{20736} \leq T_4(1) \leq 1.$$

Proof. For $f \in \mathcal{K}_{\cos}$, we have

$$1 + \frac{zf''(z)}{f'(z)} = \cos(w(z)), \text{ for all } z \in \mathbb{D},$$

where w is the Schwarz function. Since $p(z) = (1 + w(z))/(1 - w(z)) \in \mathcal{P}$, we get

$$1 + \frac{zf''(z)}{f'(z)} = \cos\left[\frac{p(z) - 1}{p(z) + 1}\right]. \tag{13}$$

Since $f(z) = z + \sum_{n=2}^{\infty} a_nz^n$, it follows that

$$\begin{aligned}\frac{zf''(z)}{f'(z)} = {} & 2a_2z + (6a_3 - 4a_2^2)z^2 + 2(4a_2^3 - 9a_2a_3 + 6a_4)z^3 \\ & - 2(8a_2^4 - 24a_2^2a_3 + 16a_2a_4 + 9a_3^2 - 10a_5)z^4 \\ & + 2(16a_2^5 - 60a_2^3a_3 + 40a_2^2a_4 + 45a_2a_3^2 - 25a_2a_5 \\ & - 30a_3a_4 + 15a_6)z^5 + \cdots \end{aligned} \tag{14}$$

Using (4), (13) and (14), we obtain

$$a_2 = 0, \tag{15}$$

$$a_3 = -\frac{p_1^2}{48}, \tag{16}$$

$$a_4 = \frac{1}{96} p_1(p_1^2 - 2p_2), \tag{17}$$

$$a_5 = \frac{1}{480}\left(-2{p_1}^4 + 9{p_1}^2 p_2 - 6p_1p_3 - 3{p_2}^2\right). \tag{18}$$

In view of (15) and (16), the third order Hermitian–Toeplitz determinant simplifies to

$$T_3(1) = 2\mathrm{Re}(a_2^2\bar{a}_3) - 2|a_2|^2 - |a_3|^2 + 1 = 1 - \frac{|p_1|^4}{2304}.$$

It is easy to verify that the subclasses $\mathcal{K}_{\cos}$ and $\mathcal{P}$ are rotationally invariant. Thus, we have $0 \le p_1 \le 2$. Next, set $p^2 =: x \in [0,4]$ such that $T_3(1) = 1 - x^2/2304$ for all $x \in [0,4]$. Using the second derivative test, we get the minimum and maximum value of $T_3(1)$. The lower bound is sharp for the function f_4 defined by

$$1 + \frac{zf_4''(z)}{f_4'(z)} = \cos z,$$

or, equivalently,

$$f_4(z) = z - \frac{1}{12}z^3 + \frac{1}{120}z^5 + \cdots, \tag{19}$$

and upper bound is sharp for the function f_5 defined by

$$1 + \frac{zf_5''(z)}{f_5'(z)} = \cos z^2,$$

or, equivalently,

$$f_5(z) = z - \frac{1}{40}z^5 + \cdots.$$

Further, substituting the values of a_2, a_3 and a_4 from (15), (16) and (17) in expression (1), we get

$$T_4(1) = 1 + \frac{|p_1|^8}{5308416} - \frac{|p_1|^4}{1152} - \frac{1}{9216}|p_1|^2|p_1^2 - 2p_2|^2. \tag{20}$$

From expressions (11) and (20), we obtain

$$T_4(1) = 1 + \frac{1}{5308416}p_1^8 - \frac{1}{1152}p_1^4 - \frac{1}{9216}p_1^2(4 - p_1^2)^2|\xi|^2.$$

Next, consider $p^2 =: x \in [0,4]$ and $|\xi| =: y \in [0,1]$ such that

$$T_4(1) = 1 + \frac{1}{5308416}x^4 - \frac{x^2}{1152} - \frac{1}{9216}x(4-x)^2y^2 = H(x,y).$$

By the second derivative test, in the region $[0,4]\times[0,1]$, the maximum value of $H(x,y)$ is 1 and the minimum value of $H(x,y)$ is 20449/20736, which give the required estimates on $T_4(1)$. The lower bound on $T_4(1)$ is the best possible for the function f_4 defined by (19) and the upper bound on $T_4(1)$ is the best possible for the function f_6 defined by

$$1+\frac{zf_6''(z)}{f_6'(z)}=\cos z^3,$$

or, equivalently,

$$f_6(z)=z-\frac{1}{84}z^7+\cdots .$$

□

Since $f\in\mathcal{K}_{\cos}$, in view of (12) and (15), we have $A_2=0$, $A_3=-a_3$ and $A_4=-a_4$. Therefore, for inverse coefficients, the third and the fourth order Hermitian–Toeplitz determinants are given by

$$T_3(1)(f^{-1})=1-|A_3|^2=T_3(1)$$

and

$$T_4(1)(f^{-1})=1+|A_3|^4-2|A_3|^2-|A_4|^2=T_4(1),$$

respectively. Thus, for the functions $f\in\mathcal{K}_{\cos}$, the invariance property holds for Hermitian–Toeplitz determinants of the third and the fourth order involving initial coefficients and inverse coefficients, respectively.

Corollary 3 *For $f\in\mathcal{K}_{\cos}$, it holds*

$$\frac{143}{144}\leq T_3(1),T_3(1)(f^{-1})\leq 1 \quad \textit{and} \quad \frac{20449}{20736}\leq T_4(1),T_4(1)(f^{-1})\leq 1.$$

3 Hankel determinants

Next, we determine bounds on $H_2(3)$ and $H_3(1)$ for the functions f from classes $\mathcal{S}^*_{\cos}$ and $\mathcal{K}_{\cos}$. To demonstrate results, the following lemmas are needed.

Lemma 3 [29, Lemma 2.3, p. 507] *Let $p\in\mathcal{P}$. Then for all $n,m\in\mathbb{N}$,*

$$|\mu p_n p_m - p_{m+n}|\leq \begin{cases} 2, & 0\leq\mu\leq 1;\\ 2|2\mu-1|, & \textit{otherwise.}\end{cases}$$

If $0<\mu<1$, then the inequality is sharp for the function $p(z)=(1+z^{m+n})/(1-z^{m+n})$. In other cases, the inequality is sharp for the function $p_0(z)=(1+z)/(1-z)$.

Lemma 4 [15] *Let $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \in \mathcal{P}$. Then, for any real number μ, we have*

$$|\mu p_3 - p_1^3| \leq \begin{cases} 2|\mu - 4|, & \mu \leq 4/3; \\ 2\mu\sqrt{\mu/(\mu-1)}, & \mu > 4/3. \end{cases}$$

The result is sharp. If $\mu \leq 4/3$, the equality holds for the function $p_0(z) := (1+z)/(1-z)$, and if $\mu > 4/3$, the equality holds for the function

$$p_1(z) := \frac{1 - z^2}{z^2 - 2\sqrt{\mu/(\mu-1)}\, z + 1}.$$

Theorem 5 *Let $f \in \mathcal{S}^*_{\cos}$. Then*

$$|H_2(3)| \leq \frac{1}{576}(46 + 27\sqrt{2}) \approx 0.146152.$$

Proof. Substituting the values of a_i's from (6)–(8) in the expression of $H_2(3)$, we get

$$H_2(3) = \frac{1}{4608} p_1^2 \left(-2p_1^4 + 5p_1^2 p_2 + 18 p_1 p_3 - 23 p_2^2\right).$$

Next, we rearrange the terms as

$$4608 H_2(3) = 23 p_1^2 p_2 \gamma_1(p_1, p_2) + 2p_1^3 \gamma_2(p_1, p_3) \tag{21}$$

where

$$\gamma_1(p_1, p_2) = \frac{5}{23} p_1^2 - p_2 \quad \text{and} \quad \gamma_2(p_1, p_3) = 9p_3 - p_1^3.$$

By Lemma 3 and Lemma 4, we get $|\gamma_1(p_1, p_2)| \leq 2$ and $|\gamma_2(p_1, p_3)| \leq 27/\sqrt{2}$. Using the triangle inequality in expression (21), bounds on $|\gamma_1(p_1, p_2)|$, $|\gamma_2(p_1, p_3)|$, and the fact that $|p_n| \leq 2$ for all $n \in \mathbb{N}$, we obtain the desired estimate on $|H_2(3)|$. □

Theorem 6 *Let $f \in \mathcal{K}_{\cos}$. Then*

$$|H_2(3)| \leq \frac{17}{480} \approx 0.0354167.$$

Proof. Substituting the values of a_i's from (16)–(18) in expression for $H_2(3)$, we get

$$H_2(3) = -\frac{1}{23040} p_1^2 \left(5p_1^4 - 19 p_1^2 p_2 + 6 p_1 p_3 + 13 p_2^2\right).$$

By rearranging the terms and using the triangle inequality, we obtain

$$23040 |H_2(3)| \leq 5|p_1^6| + 6|p_1^3 p_3| + 13 |p_1^2 p_2| |\gamma_3(p_1, p_2)| \tag{22}$$

where

$$\gamma_3(p_1,\, p_2) = \frac{19}{13}p_1^2 - p_2. \tag{23}$$

Using Lemma 3, we get

$$|\gamma_3(p_1,\, p_2)| \leq 50/13. \tag{24}$$

From (22), (24) and $|p_n| \leq 2$ for all $n \in \mathbb{N}$, we obtain the desired bound on $|H_2(3)|$. □

Theorem 7 *Let $f \in \mathcal{S}^*_{\cos}$. Then*

$$|H_3(1)| \leq \frac{23}{288} + \frac{3}{4\sqrt{137}} \approx 0.143938.$$

Proof. Since $f \in \mathcal{S}^*_{\cos}$, $a_2 = 0$, and hence, $H_3(1) = -a_3^3 - a_4^2 + a_3a_5$. Substituting the values of a_i's from (6)–(8), we get

$$H_3(1) = \frac{1}{36864}\left(-7p_1^6 + 40p_1^4p_2 - 184p_1^2p_2^2 + 144p_1^3p_3\right).$$

Next, we rearrange the terms as

$$36864H_3(1) = 7p_1^3\gamma_4(p_1,\, p_3) + 184p_1^2p_2\gamma_5(p_1,\, p_2)$$

where

$$\gamma_4(p_1,\, p_3) = \frac{144}{7}p_3 - p_1^3 \quad \text{and} \quad \gamma_5(p_1,\, p_2) = \frac{5}{23}p_1^2 - p_2.$$

Using Lemma 3 and Lemma 4, we get

$$|\gamma_4(p_1,\, p_3)| \leq 3456/7\sqrt{137} \qquad \text{and} \qquad |\gamma_5(p_1,\, p_2)| \leq 2. \tag{25}$$

Using the triangle inequality, inequalities (25) and $|p_n| \leq 2$ for all $n \in \mathbb{N}$, we get the desired estimate. □

Theorem 8 *Let $f \in \mathcal{K}_{\cos}$. Then*

$$|H_3(1)| \leq \frac{301}{8640} \approx 0.034838.$$

Proof. Since $f \in \mathcal{K}_{\cos}$, $H_3(1) = -a_3^3 - a_4^2 + a_3a_5$. Substituting the values of a_i's from (16)–(18), we get

$$H_3(1) = \frac{1}{552960}\left(-115p_1^6 + 456p_1^4p_2 - 312p_1^2p_2^2 - 144p_1^3p_3\right).$$

By rearranging the terms and using the triangle inequality, we get

$$552960|H_3(1)| \leq 115|p_1^6| + 144|p_1^3p_3| + 312|p_1^2p_2||\gamma_3(p_1,\, p_2)| \tag{26}$$

where $\gamma_3(p_1,\, p_2)$ is given by (23). In view of inequalities (26), (24) and $|p_n| \leq 2$ for all $n \in \mathbb{N}$, we get $552960|H_3(1)| \leq 19264$. □

Acknowledgements. The authors are thankful to the referees and the editors for the insightful comments which improved this manuscript. Both authors would like to thank the Institute of Eminence, University of Delhi, Delhi, India–110007 for providing financial support for this research under the grant IoE/2021/12/FRP.

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Please, cite to this paper as published in
Armen. J. Math., V. **16**, N. 9(2024), pp. 1–18
https://doi.org/10.52737/18291163-2026.16.09-1-18