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Characterization of the Three-Variate Inverted Dirichlet Distributions

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Abstract. In this paper, we prove a characterization of three-variate inverted Dirichlet distributions by an independence property. The main technical challenge was a problem involving the solution of a related functional equation.

Key Words: Characterization of Probability Distributions, Functional Equation, Independence, Inverted Dirichlet Distribution, Transformation
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Introduction

In the statistics literature, several authors have proved characterizations of probability distributions using the independence of transformations. For example, let X and Y be independent non-degenerate positive random variables. Lukacs [7] proved that the random variables $U = X/Y$ and $V = X + Y$ are independent if and only if X and Y have gamma distributions. Letac and Wesolowski [6] used the transformation $U = 1/(X + Y)$ and $V = 1/X - 1/(X + Y$ and proved that U and V are independent if and only if X has the generalized inverse Gaussian distribution and Y has the gamma distribution. Wesolowski [10] considered the transformation $(U, V) = (1 - XY, (1 - X)/(1 - XY))$, and proved that U and V are independent if and only if X and Y follow beta distributions.

Darroch and Ratcliff [4] proved that if $X_1, ..., X_n$ are positive random variables with continuous probability density functions such that $\sum_{i=1}^{n} X_i < 1$, then, for every $i \in \{1, ..., n\}$, $X_i/(1 - \sum_{j \neq i} X_j)$ is independent of the set $\{X_j; j \neq i\}$ if and only if the vector $(X_1, ..., X_n)$ has a Dirichlet distribution.

Extensions of these characterizations on symmetric matrices have also been studied (see, for example, Olkin and Rubin [8], Letac and Wesolowski [6], Kolodziejek [5] and Ben Farah and Hassairi [3]).

A random vector (X_1, X_2, X_3) is said to have three-variate inverted Dirichlet distribution with positive parameters p_1, p_2, p_3 and p_4, denoted by $\mathcal{ID}(p_1, p_2, p_3; p_4)$ if probability density function is given by

$$\frac{\Gamma(p)}{\prod_{i=1}^{4}\Gamma(p_i)}\prod_{i=1}^{3} x_i^{p_i-1}\left(1+\sum_{i=1}^{3} x_i\right)^{-p},$$

where $x_i > 0$, $i = 1, 2, 3$ and $p = \sum_{i=1}^{4} p_i$.

The inverted Dirichlet distribution has many interesting properties (see, for example, Tiao and Guttman [9] and Bdiri and Bouguila [1]). We mention that Ben Farah [2] studied a generalization of this distribution on symmetric matrices.

Define two transformations

$$\Psi_1(x,y,z) = \left(x+y, \frac{x}{x+y}, \frac{z}{1+x+y}\right) \in (0,\infty)\times(0,1)\times(0,\infty), \quad (1)$$

and

$$\Psi_2(x,y,z) = \left((x,z), \frac{y}{1+x+z}\right) \in (0,\infty)^2\times(0,\infty), \quad (2)$$

where $(x,y,z) \in (0,\infty)^3$. The aim of this paper is to prove that the independence of the components of Ψ_1 and that of Ψ_2 characterizes the three-variate inverted Dirichlet distributions. The proof is based on a solution of a related functional equation. This equation is solved, under technical smoothness conditions, in Section 1.

1 Functional equation

In this section, we solve the functional equation which was essential for proving the characterization derived in the next section.

Theorem 1 *Let g_i, $i = 1, \ldots, 5$, be continuously differentiable functions satisfying*

$$g_1(x,z) + g_2\left(\frac{y}{1+x+z}\right) = g_3(x+y) + g_4\left(\frac{x}{x+y}\right) + g_5\left(\frac{z}{1+x+y}\right), \quad (3)$$

for any $x, y, z \in (0,\infty)$. Then there exist constants $\alpha, \beta, \gamma, \delta, \theta, \eta$ and c_i, $i = 1, \ldots, 5$ such that $\theta - \eta - \alpha = 0$ and

$$\begin{aligned}
g_1(x,z) &= \ln\left(c_1 x^{-\beta-\delta} z^{\eta}(1+x+z)^{\alpha+\beta}\right),\\
g_2(x) &= \ln\left(c_2 x^{\beta}(1+x)^{\alpha}\right),\\
g_3(x) &= \ln\left(c_3 x^{-\delta}(1+x)^{\theta}\right),\\
g_4(x) &= \ln\left(c_4 x^{-\beta-\delta}(1-x)^{\beta}\right),\\
g_5(x) &= \ln\left(c_5 x^{\eta}(1+x)^{\alpha}\right).
\end{aligned}$$

Proof. Define new variables a, b and c as

$$(a, b, c) = \Psi_1(x, y, z),$$

where Ψ_1 defined in (1). Then

$$(x, y, z) = (ab, a(1-b), (1+a)c),$$

and (3) takes the form

$$g_1(ab, (1+a)c) + g_2\left(\frac{a(1-b)}{1+ab+(1+a)c}\right) = g_3(a) + g_4(b) + g_5(c). \qquad (4)$$

Taking the derivatives of (4) with respect to a, b and c yields the following three equations:

$$b\frac{\partial g_1}{\partial x_1}(ab, (1+a)c) + c\frac{\partial g_1}{\partial x_2}(ab, (1+a)c) + \frac{(1-b)(1+c)}{(1+ab+(1+a)c)^2} \times g_2'\left(\frac{a(1-b)}{1+ab+(1+a)c}\right) = g_3'(a),$$

$$a\frac{\partial g_1}{\partial x_1}(ab, (1+a)c) - \frac{a(1+a)(1+c)}{(1+ab+(1+a)c)^2} g_2'\left(\frac{a(1-b)}{1+ab+(1+a)c}\right) = g_4'(b),$$

$$(1+a)\frac{\partial g_1}{\partial x_2}(ab, (1+a)c) - \frac{a(1+a)(1-b)}{(1+ab+(1+a)c)^2} \times g_2'\left(\frac{a(1-b)}{1+ab+(1+a)c}\right) = g_5'(c).$$

Eliminating from the above three equations the partial derivatives $\frac{\partial g_1}{\partial x_1}$ and $\frac{\partial g_1}{\partial x_2}$, we get

$$\frac{a}{1+ab+(1+a)c} g_2'\left(\frac{a(1-b)}{1+ab+(1+a)c}\right) = a g_3'(a) - b g_4'(b) - \frac{ac}{1+a} g_5'(c). \qquad (5)$$

Since the limit of the left side of (5) as c tends to 0 exists, the limit $\lim_{c \longrightarrow 0} c g_5'(c) = T$ also exists. Thus, we get

$$\frac{a}{1+ab} g_2'\left(\frac{a(1-b)}{1+ab}\right) = a g_3'(a) - b g_4'(b) - \frac{a}{1+a} T,$$

which can be rewritten as

$$\frac{a}{1+a}\frac{1+a}{1+ab}g_2'\left(\frac{1+a}{1+ab}-1\right)=ag_3'(a)-bg_4'(b)-\frac{a}{1+a}T. \tag{6}$$

Denote

$$\begin{aligned} g(x) &= xg_2'(x-1), \\ h_1(x) &= xg_3'(x)-\frac{x}{1+x}T, \\ h_2(x) &= -xg_4'(x). \end{aligned} \tag{7}$$

Then (6) takes the form

$$\frac{a}{1+a}g\left(\frac{1+a}{1+ab}\right)=h_1(a)+h_2(b),$$

where $a>0$ and $0<b<1$. For $y=\dfrac{a}{1+a}\in(0,1)$ and $t=1-b\in(0,1)$, from the above equation we obtain

$$yg\left(\frac{1}{1-ty}\right)=h_1\left(\frac{y}{1-y}\right)+h_2(1-t).$$

Put

$$\begin{aligned} G(x) &= g\left(\frac{1}{1-x}\right), \\ H_1(x) &= h_1\left(\frac{x}{1-x}\right), \\ H_2(x) &= h_2(1-x). \end{aligned} \tag{8}$$

Then we obtain

$$yG(ty)=H_1(y)+H_2(t), \qquad 0<y,t<1. \tag{9}$$

Letting $t\to 1$ in (9), we get

$$H_1(y)=yG(y)-K, \tag{10}$$

and letting $y\to 1$ in (9), we get

$$H_2(t)=G(t)-L, \tag{11}$$

where $K=H_2(1)$ and $L=H_1(1)$. Putting all of the above back into (9) and multiplying both sides of the obtained equation by t, we get

$$tyG(ty)=tyG(y)+tG(t)+tA, \tag{12}$$

where $A=-K-L$. Let

$$\varphi(x)=xG(x)+A. \tag{13}$$

We can rewrite (12) as

$$\varphi(ty) = t\varphi(y) + \varphi(t).$$

Taking in the above equation $t = \frac{1}{2}$ and $y = \frac{1}{2}$, we obtain the following two equations:

$$\begin{aligned}\varphi\left(\frac{y}{2}\right) &= \frac{1}{2}\varphi(y) + \varphi\left(\frac{1}{2}\right),\\ \varphi\left(\frac{t}{2}\right) &= t\varphi\left(\frac{1}{2}\right) + \varphi(t).\end{aligned}$$

Hence, taking $t = y$, we get

$$\varphi(y) = \alpha y - \alpha, \qquad y \in (0,1),$$

where $\alpha = -2\varphi\left(\frac{1}{2}\right)$. Hence, by (13),

$$xG(x) = \alpha x + \beta,$$

which implies

$$G(x) = \alpha + \frac{\beta}{x},$$

where $\beta = -\alpha + K + L$.

By (10) and (11), we get

$$\begin{aligned}H_1(y) &= \alpha y - \alpha + L,\\ H_2(c) &= \frac{\beta}{c} + \alpha - L.\end{aligned}$$

From (8), we obtain

$$\begin{aligned}g(x) &= \frac{\beta}{x-1} - \alpha + \beta,\\ h_1(x) &= -\frac{\alpha}{1+x} + L,\\ h_2(x) &= -\frac{\beta}{1-x} + \delta,\end{aligned}$$

where $\delta = \alpha - L$. Now (7) yields

$$\begin{aligned}g_2'(x) &= \frac{\beta}{x} + \frac{\alpha}{1+x},\\ g_3'(x) &= -\frac{\delta}{x} + \frac{\theta}{1+x},\\ g_4'(x) &= -\frac{\beta+\delta}{x} - \frac{\beta}{1-x},\end{aligned}$$

where $\theta = \alpha + T$.

Letting $b \to 0$ in (5), we obtain

$$\frac{a}{1+(1+a)c} g_2' \left(\frac{a}{1+(1+a)c} \right) = a g_3'(a) - \frac{ac}{1+a} g_5'(c).$$

Inserting $a = 1$, we arrive at

$$\frac{1}{1+2c} g_2' \left(\frac{1}{1+2c} \right) = g_3'(1) - \frac{c}{2} g_5'(c).$$

Hence,

$$\begin{aligned} g_5'(x) &= \frac{2}{x} g_3'(1) - \frac{2}{x} \frac{1}{1+2x} g_2' \left(\frac{1}{1+2x} \right) \\ &= \frac{\eta}{x} + \frac{\alpha}{1+x}, \end{aligned}$$

where $\eta = 2g_3'(1) - \alpha - 2\beta$.

Thus,

$$\begin{aligned} g_2(x) &= \ln(c_2 x^{\beta}(1+x)^{\alpha}), \\ g_3(x) &= \ln(c_3 x^{-\delta}(1+x)^{\theta}), \\ g_4(x) &= \ln(c_4 x^{-\beta-\delta}(1-x)^{\beta}), \\ g_5(x) &= \ln(c_5 x^{\eta}(1+x)^{\alpha}). \end{aligned}$$

Moreover, from (3), it follows that

$$g_1(x,z) = \ln \left(c_1 x^{-\beta-\delta} z^{\eta} (1+x+z)^{\alpha+\beta} (1+x+y)^{\theta-\eta-\alpha} \right),$$

and thus, $\theta - \eta - \alpha = 0$. □

2 Characterization

In this section, we state and prove our main characterization result.

Theorem 2 *Let (X,Y,Z) be a random vector with strictly positive, continuously differentiable density. Then the random vector (X,Z) and $\dfrac{Y}{1+X+Z}$ are independent and the random vector $\left(X+Y, \dfrac{X}{X+Y}, \dfrac{Z}{1+X+Y} \right)$ has independent components if and only if (X,Y,Z) has a three-variate inverted Dirichlet distribution.*

Proof. The necessity is trivial to prove therefore we will prove the sufficiency-half of the theorem.

Let $(u_1, u_2, u_3) = \Psi_1(x,y,z)$, where Ψ_1 is defined in (1). The Jacobian of Ψ_1 is

$$\mid J_1 \mid = \frac{1}{(x+y)(1+x+y)}.$$

Thus, using the independence of U_i, $i = 1, 2, 3$, the density f of (X, Y, Z) can be expressed as

$$f(x,y,z) = \frac{f_{U_1}(x+y)}{(x+y)(1+x+y)} f_{U_2}\left(\frac{x}{x+y}\right) f_{U_3}\left(\frac{z}{1+x+y}\right). \qquad (14)$$

Similarly, let $(v_1, v_2, v_3) = \Psi_2(x, y, z)$, where Ψ_2 is defined in (2). The Jacobian of Ψ_2 is

$$\mid J_2 \mid = \frac{1}{1+x+z}.$$

Thus, using the independence of (V_1, V_2) and V_3, the density f of (X, Y, Z) can be expressed as

$$f(x,y,z) = \frac{1}{1+x+z} f_{(V_1,V_2)}(x,z) f_{V_3}\left(\frac{y}{1+x+z}\right), \qquad (15)$$

Combining (14) and (15) we have

$$\begin{aligned}\frac{1}{(x+y)(1+x+y)} f_{U_1}(x+y) f_{U_2}\left(\frac{x}{x+y}\right) f_{U_3}\left(\frac{z}{1+x+y}\right) \\ = \frac{1}{1+x+z} f_{(V_1,V_2)}(x,z) f_{V_3}\left(\frac{y}{1+x+z}\right). \qquad (16)\end{aligned}$$

Let

$$\begin{aligned} g_1(x,z) &= \ln\left(\frac{1}{1+x+z} f_{(V_1,V_2)}(x,z)\right), \\ g_2(x) &= \ln\left(f_{V_3}(x)\right), \\ g_3(x) &= \ln\left(\frac{1}{x(1+x)} f_{U_1}(x)\right), \\ g_4(x) &= \ln(f_{U_2}(x)), \\ g_5(x) &= \ln(f_{U_3}(x)). \end{aligned}$$

Then, we can rewrite (16) as

$$g_1(x,z) + g_2\left(\frac{y}{1+x+z}\right) = g_3(x+y) + g_4\left(\frac{x}{x+y}\right) + g_5\left(\frac{z}{1+x+y}\right).$$

Note that the above equation is the one we solved in Theorem 1. Hence, we conclude that there exist constants p_1, p_2, p_3, p_4 such that

$$\begin{aligned} f_{(V_1,V_2)}(x,z) &= c_1 x^{p_1-1} z^{p_3-1}(1+x+z)^{-p_1-p_3-p_4}, \\ f_{V_3}(x) &= c_2 x^{p_2-1}(1+x)^{-p_1-p_2-p_3-p_4}, \\ f_{U_1}(x) &= c_3 x^{p_1+p_2-1}(1+x)^{-p_1-p_2-p_4}, \\ f_{U_2}(x) &= c_4 x^{p_1-1}(1-x)^{p_2-1}, \\ f_{U_3}(x) &= c_5 x^{p_3-1}(1+x)^{-p_1-p_2-p_3-p_4}, \end{aligned}$$

where the c_i's are some normalizing constants. The integrability condition implies that p_1, p_2, p_3 and p_4 are positive.

Thus, from (14), it follows that

$$f(x,y,z) = kx^{p_1-1}y^{p_2-1}z^{p_3-1}(1+x+y+z)^{-p_1-p_2-p_3-p_4}.$$

Hence, $(X,Y,Z) \sim \mathcal{ID}_3(p_1,p_2,p_3;p_4)$. $\square$

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