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# A Note on Omitting Types in Propositional Logic

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**Abstract.** Analogues of the classical omitting types theorems of first-order logic are proved for propositional logic. For an infinite cardinal $\kappa$, a sufficient criterion is given for the omission of $\kappa$-many types in a propositional language with $\kappa$ propositional variables.

*Key Words:* propositional logic, type, omitting types theorem
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# Introduction

Classical omitting types theorems are important results in the model theory of first-order and higher order logics; see [2] and [3]. Their proofs are frequently assimilated to variants of model-theoretic forcing or game-theoretic arguments. The theorems themselves are widely employed in the construction of e.g. non-standard models of Peano Arithmetic and models of first-order theories, including Zermelo-Frankel set theory. However, the concept of a type makes sense in propositional logic and it is natural to explore whether analogues of the omitting types results can be proved in this simpler setting. Theorem 1 answers the case where the family of propositional types is countable, while Theorem 2, the main result of this note, provides a sufficient criterion for the omission of $\kappa$-many types in a propositional language with $\kappa$ propositional variables for an infinite cardinal $\kappa$.

In the remainder of this section, we fix notation and recall for convenience the pertinent definitions of models, types, and type omission, both local and global. The reader may wish to skip directly to the main section of the paper, where the principal results are stated and proved.

Following [1], let $L$ be a non-empty collection of propositional variables (sentence symbols). An $L$-sentence (or proposition) is a Boolean combination of propositional variables, using the propositional operations $\wedge, \vee, \neg$, with the abbreviation $\varphi \to \psi$ meaning $\neg\varphi \vee \psi$. A *model* for $L$ (an $L$-model or an $L$-valuation) is a subset $A \subseteq L$; the intuitive meaning is that $A$ assigns the truth value $\top$ (for "true") to a propositional variable $p$ if $p \in A$ (and $\perp$ otherwise), in other words, $A \in {}^{L}\{\top, \perp\}$, the family of functions from $L$ into $\{\top, \perp\}$, and $A(p) = \top$ is a restatement of $p \in A$. We write $A \models \varphi$ to mean that the $L$-sentence $\varphi$ is true in the model $A$ under the standard inductive definition on the complexity of the $L$-sentence $\varphi$. For a family $\Sigma$ of $L$-sentences, the notation $\Sigma \vdash \varphi$ means there is a deduction of $\varphi$ from $\Sigma$; $\Sigma$ is *inconsistent* if $\Sigma \vdash \varphi$ for every $L$-sentence $\varphi$, equivalently $\Sigma \models \neg\varphi \wedge \varphi$ for some $L$-sentence $\varphi$; otherwise $\Sigma$ is *consistent*. We write $A \models \Sigma$ if $A \models \varphi$ for every $\varphi \in \Sigma$, and, for a family $T$ of $L$-sentences, $T \models \Sigma$ if every model of $T$ is also a model of $\Sigma$. The family $\Sigma$ is *satisfiable* if $\Sigma$ has at least one model $A$. An *$L$-theory* is a family $T$ of $L$-sentences; $T$ is *complete* if for every $L$-sentence $\varphi$, $T \models \varphi$ or $T \models \neg\varphi$. A *propositional theory* is just an $L$-theory for some family $L$ of propositional variables. We write $\bigwedge \Gamma$ for the conjunction of a finite family $\Gamma$ of $L$-sentences. Our set-theoretic notation is standard: $\omega$ is the first infinite ordinal, $\alpha, \beta, \ldots, \xi, \ldots$ are ordinals, $\aleph_1$ is the first uncountable cardinal. A set $Y$ is *countable* if $Y$is a surjective image of $\omega$. We use $|X|$ to denote the cardinality of a set $X$; $2^{\aleph_0}$ is the cardinality of the power set $P(\omega) = \{Z : Z \subseteq \omega\}$ of $\omega$; $\kappa, \lambda, \ldots$ denote infinite cardinals, and $[\kappa]^\lambda$ is the set of subsets of $\kappa$ of cardinality $\lambda$.

We turn to the specific concepts required for the statement of the propositional omitting types theorem.

**Definition 1** *A* propositional type *in $L$ (or simply an $L$-type or type) is a set $\Sigma$ of $L$-sentences.*

**Definition 2** *An $L$-model $A$* realises *the type $\Sigma$ if $A \models \Sigma$; otherwise $A$* omits *$\Sigma$.*

So for the purposes of the present note, propositional types and propositional theories are the same thing (although some sources require theories to be deductively closed); this contrasts with first-order logic, where a type is a set of formulas with a tuple of free variables, while a theory is a collection of sentences, i.e. formulas in which every variable is quantified (bound).

**Definition 3** *A propositional $L$-theory $T$* omits *the type $\Sigma$ if every model of $T$ omits $\Sigma$. The theory $T$* locally realises *the type $\Sigma$ if for some $L$-sentence $\varphi$, $T \cup \{\varphi\}$ is consistent, and for every $\sigma \in \Sigma, T \models \varphi \to \sigma$. Otherwise, $T$* locally omits *$\Sigma$.*

# Omitting propositional types theorems

With the definitions in hand, let us prove some very simple propositional omitting types results.

**Proposition 1** *If $T$ is a complete consistent propositional theory and $T$ omits the propositional type $\Sigma$, then $T$ locally omits $\Sigma$.*

***Proof.*** Suppose $T$ does not locally omit $\Sigma$. Then there is an $L$-sentence $\varphi$ such that (1) $T \cup \{\varphi\}$ is consistent, and (2) for every $\sigma \in \Sigma, T \models \varphi \to \sigma$. Let $A \models T$. Since $T$ is complete, $A \models T \cup \{\varphi\}$ by (1) and so by (2), $A \models \sigma$ for every $\sigma \in \Sigma$, in other words $T$ does not omit $\Sigma$. □

The classical omitting types theorem is a converse of the first-order version of the previous result.

**Proposition 2** *If $T$ is a consistent propositional theory and $T$ locally omits the propositional type $\Sigma$, then $T$ has a model which omits $\Sigma$.*

***Proof.*** Suppose the conclusion fails and $T \models \Sigma$. Then for any (some) $L$-sentence $\varphi, T \cup \{\varphi \vee \neg\varphi\} \models \Sigma$ and so $T \models (\varphi \vee \neg\varphi) \to \sigma$ for every $\sigma \in \Sigma$. However, $T$ locally omits $\Sigma$, hence $T \models \neg(\varphi \vee \neg\varphi)$, which means $T \models \neg\varphi \wedge \varphi$, in other words $T$ is inconsistent. □

The Extended Omitting Types theorem for countably many propositional types is due to Thomas Forster (in unpublished correspondence). Since we will consider shortly counterexamples and appropriate generalisations to uncountable families of propositional types, we state and prove it next.

**Theorem 1** (Omitting countable families of propositional types) *If $T$ is a consistent propositional theory and $T$ locally omits the propositional types $\Sigma_n$ for $n < \omega$, then $T$ has a model which omits every $\Sigma_n$.*

***Proof.*** Claim: $(\forall n \in \omega)(\exists \sigma_n \in \Sigma_n)$ such that $(*)_n$ $T \cup \langle \neg\sigma_k : k \leq n\rangle$ is consistent. For $n = 0$, this follows directly from the previous proposition on omitting a single type: $T$ locally omits $\Sigma_0$ and hence has a model $A$ omitting $\Sigma_0$; let $\sigma_0 \in \Sigma_0$ be such that $A \models \neg\sigma_0$, so $(*)_0$ is satisfied. Given $\langle \sigma_k : k \leq n\rangle$, let $A \models T$ witness $(*)_n$, and let $\varphi$ be $\bigwedge_{k \leq n} \neg\sigma_k$, so $(**) A \models \varphi$. If no $\sigma \in \Sigma_{n+1}$ satisfies $(*)_{n+1}$, then $T \cup \{\varphi\} \models \Sigma_{n+1}$, and so $T \models \varphi \to \sigma$ for every $\sigma \in \Sigma_{n+1}$. Since $T$ locally omits $\Sigma_{n+1}$, $T \models \neg\varphi$. But $A \models T$, so $A \models \bigvee_{k \leq n} \sigma_k$, contradicting $(**)$. The proof of the claim is complete.

Since $T \cup \{\neg\sigma_n : n < \omega\}$ is finitely satisfiable, it follows by the Compactness theorem, that $T \cup \{\neg\sigma_n : n < \omega\}$ has a model $A$, which clearly omits every $\Sigma_n$, as required. □

Now let us consider the size of the family of types and the question whether larger families can be omitted.

First, note that it is not in general possible to omit uncountably many types in a language with infinitely many propositional variables, as evidenced by the following straightforward proposition.

**Proposition 3** *There is a consistent theory $T$ in a countable propositional language $L$ and an uncountable family of $L$-types such that $T$ locally omits each type, but $T$ does not omit all types simultaneously.*

***Proof.*** Let $L$ have $\omega$ distinct propositional variables $P = \langle p_i : i < \omega \rangle$, and for each $L$-model $A$, let $\Gamma_A$ be the set $\{\gamma_n^A : n < \omega\}$ where $\gamma_n^A$ is the $L$-formula $\bigwedge_{i<n} q_i$ and $q_i$ is $p_i$ iff $A(p_i) = \top$ (otherwise $q_i$ is $\neg p_i$). Note that $\Gamma_A$ is an $L$-type and is consistent since $A(\gamma_n^A) = \top$ for every $n < \omega$. Let $T = \emptyset$ be the empty $L$-theory. Then every $L$-model satisfies $T$ and realises some type $\Gamma_A$. However, $T$ locally omits every $\Gamma_A$. [Suppose $\psi$ is an $L$-formula such that for all $n < \omega$, $T \models \psi \to \gamma_n^A$. We show $T \models \neg\psi$. Otherwise, there is a model $B$ satisfying $T$ such that $B(\psi) = \top$. It follows that $B = A$ since $B(\gamma_n^A) = \top$ for every $n < \omega$. Let $\langle p_i : i < n_0 \rangle$ contain all the propositional variables occurring in $\psi$. Let $C \restriction \langle p_i : i < n_0 \rangle = A \restriction \langle p_i : i < n_0 \rangle$, and $C(p_k) = A(\neg p_k)$ for $k \geq n_0$. Then $C$ is a model satisfying $T \cup \{\psi\}$, and so for all $m$, $C(\psi \to \gamma_m^A) = \top$, which implies $C(\gamma_m^A) = \top$. Now taking $m \geq n_0$ yields an immediate contradiction.] $\Box$

One can refine this example by adding a family $Q$ of new propositional variables and taking $T$ to any consistent theory all of whose sentences have propositional variables only amongst $Q$.

Whether an example can be devised that involves omitting exactly $\aleph_1$ types whatever the set theory is unclear. Under the Continuum Hypothesis $2^{\aleph_0} = \aleph_1$, the above example settles the question completely.

Next we examine a generalisation of the Extended Omitting Propositional Types theorem, following closely [3].

**Lemma 1** *Let $L$ have at most $\kappa$ propositional variables, $T$ be a consistent $L$-theory, and for each $\xi < \kappa$, $\Gamma_\xi$ be an $L$-type.*

*If for each $\xi < \kappa$, whenever $\Sigma$ is a set of $L$-formulas such that $|\Sigma| < \kappa$ and $T \cup \Sigma$ is consistent, there exists $\gamma \in \Gamma_\xi$ such that $T \cup \Sigma \cup \{\neg\gamma\}$ is consistent, then $T$ has an $L$-model that omits every $\Gamma_\xi$ for $\xi < \kappa$.*

***Proof.*** Let $\langle \varphi_\alpha : \alpha < \kappa \rangle$ enumerate all the $L$-formulas. Define an increasing sequence $\langle T_\alpha : \alpha < \kappa \rangle$ of $L$-theories as follows:

1. $T_{\alpha+1}$ is a finite consistent extension of $T_\alpha$;

2. $T_\delta = \cup_{\alpha<\delta} T_\alpha$ whenever $\delta$ is a limit ordinal;

3. either $\varphi_\alpha \in T_{\alpha+1}$ or $\neg\varphi_\alpha \in T_{\alpha+1}$.

Let $T_0 = T$. Suppose $T_\alpha$ has been defined; let $\Sigma = T_\alpha \setminus T$. Since $T \cup \Sigma$ is consistent, $|\Sigma| < \kappa$, and $\alpha < \kappa$, there exists $\gamma \in \Gamma_\alpha$ such that $T \cup \Sigma \cup \{\neg\gamma\}$ is consistent. Put $\neg\gamma$ in $T_{\alpha+1}$. If $T_\alpha \cup \{\neg\gamma\} \cup \{\varphi_\alpha\}$ is consistent, add $\varphi_\alpha$ to

$T_{\alpha+1}$; otherwise add $\neg\varphi_\alpha$ to $T_{\alpha+1}$. The theory $T_\kappa = \cup_{\alpha<\kappa} T_\alpha$ is a maximal consistent set of $L$-formulas and hence possesses a model $A$ which satisfies $T$ and omits $\Gamma_\xi$ for every $\xi < \kappa$. $\square$

**Theorem 2** (Omitting $\kappa$-many propositional types) *Let $L$ have at most $\kappa$ propositional variables, $T$ be a consistent $L$-theory, and for each $\xi < \kappa$, $\Gamma_\xi = \{\gamma_\alpha^\xi : \alpha < \kappa\}$ be an $L$-type.*

*If (1) $\alpha \leq \beta < \kappa \Rightarrow T \models \gamma_\beta^\xi \to \gamma_\alpha^\xi$ and (2) whenever $\psi$ is an $L$-formula such that $T \cup \{\psi\}$ is consistent and $\xi < \kappa$, there exists $\alpha < \kappa$ such that $T \cup \{\psi\} \cup \{\neg\gamma_\alpha^\xi\}$ is consistent, then $T$ has an $L$-model omitting every $\Gamma_\xi$.*

***Proof.*** We remark that the hypotheses of the previous result hold: for each $\xi < \kappa$, whenever $\Sigma$ is a set of $L$-formulas such that $|\Sigma| < \kappa$ and $T \cup \Sigma$ is consistent, there exists $\gamma \in \Gamma_\xi$ such that $T \cup \Sigma \cup \{\neg\gamma\}$ is consistent. In more detail: suppose there is a counterexample for some $\xi < \kappa$, with $|\Sigma| < \kappa$ and $T \cup \Sigma$ consistent. So for every $\alpha < \kappa$, $T \cup \{\psi\} \cup \{\neg\gamma_\alpha^\xi\}$ is inconsistent. Since $|\Sigma| < \kappa$, there are $I \in [\kappa[^\kappa$ and $\Sigma_0 \in [\Sigma[^{<\omega}$ such that for every $\beta \in I$, $T \cup \Sigma_0 \cup \{\neg\gamma_\beta^\xi\}$ is inconsistent. Let $\psi^*$ be $\bigwedge \Sigma_0$. By (2), there exists $\alpha < \kappa$ such that $T \cup \{\psi^*\} \cup \{\neg\gamma_\alpha^\xi\}$ is consistent. Now if $A$ is an $L$-model satisfying $T \cup \{\psi^*\} \cup \{\neg\gamma_\alpha^\xi\}$, then taking $\beta > \alpha, \beta \in I$, it follows $A(\neg\gamma_\beta^\xi) = \top$ using (1). A contradiction, since $T \cup \{\psi^*\} \cup \{\neg\gamma_\beta^\xi\}$ is inconsistent. $\square$

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