Armenian Journal of Mathematics
Volume 7, Number 1, 2015, 50–58

On the structure of C^*-algebra generated by a family of partial isometries and multipliers

A. Yu. Kuznetsova and Ye. V. Patrin
Kazan Federal University

Abstract. In the paper we consider an operator algebra generated by a family of partial isometries associated with a selfmapping on a countable set and by multipliers.
An action of the unit circle on this algebra is specified that determines its $\mathbb{Z}$-grading. Under some conditions on the mapping the algebra is isomorphic to the crossed product of its fixed point subalgebra and the semigroup $\mathbb{N}$.

Key Words: C^*-algebra, partial isometry, conditional expectation, Toeplitz algebra, crossed product, $*$-endomorphism
Mathematics Subject Classification 2000: 46L99

Introduction

In the paper we continue to study the structure of the algebra $\mathfrak{M}_\varphi$ (see [6]), generated by a family of partial isometries and by the commutative algebra of multipliers. The starting point is a selfmapping $\varphi : X \longrightarrow X$ on a countable set X with finite numbers of preimages of each point. This mapping generates a directed graph with vertices at the points of the set X and the edges $(x, \varphi(x))$.
This mapping also induces an operator $T_\varphi : l^2(X) \to l^2(X), \quad T_\varphi f = f \circ \varphi$.
A family $\mathcal{U}$ of partial isometries $U_n, n = 1, 2, \ldots$ is associated with this composition operator T_φ such that $T_\varphi = U_1 + \sqrt{2}U_2 + \cdots + \sqrt{m}U_m + \cdots$. We use the notation $\mathcal{U}^*$ for the family of partial isometries $\{U_k^*, k = 1, 2, \ldots\}$. Operators $Q_\varphi = \sum\limits_{k\in\mathbb{N}} U_k U_k^* = \sum\limits_{k\in\mathbb{N}} Q_k$ and $P_\varphi = \sum\limits_{k\in\mathbb{N}} U_k^* U_k = \sum\limits_{k\in\mathbb{N}} P_k$ are projections determined by the initial mapping.
The operator algebra generated by such family of partial isometries (finite or countable), was studied in [4, 5].

Let $B(X)$ be the algebra of all bounded functions on X. Each function f from $B(X)$ generates a multiplier operator $M_f g = fg$, $g \in l^2(X)$ such that $(M_f)^* = M_{\overline{f}}$. The C^*-algebra generated by all multipliers is denoted by $\mathrm{M}(X)$. This algebra is maximal commutative subalgebra (masa) in $B(l^2(X))$.
The main object of study in the paper is the C^*-algebra $\mathfrak{M}_\varphi$, a subalgebra of $B(l^2(X))$ generated by the algebra $\mathrm{M}(X)$ and partial isometries from $\mathcal{U}$. It was shown in [6], that $\mathfrak{M}_\varphi$ is, in particular, a nuclear $\mathbb{Z}$-graded algebra.
The algebra $\mathfrak{M}_\varphi$ can be considered as a modification of the Arzumanian-Vershik algebra ([2, 3]), which is defined as the regular representation of the algebra generated by the bicyclic semigroup and a commutative algebra with natural commutation relations.

1 Preliminaries

Here we present briefly the basic information that will be needed further. More detailed account can be found in [4, 6].
Throughout what follows some restrictions on the initial mapping are assumed.
We suppose the mapping φ to be fixed satisfying the following conditions:

(i) *there is no cyclic element in X, i.e. an element that $\varphi^n(x) = x$ for some $n \in \mathbb{N}$*

(ii) *the number of preimages is uniformly bounded, i.e. a number m exists such that*
$$m = \sup_{x \in X} \operatorname{card}\{\varphi^{-1}(x)\} < \infty \tag{1}$$

Under the last condition the operator $T = T_\varphi$ is bounded, and is a *finite* sum
$T = U_1 + \sqrt{2}U_2 + \cdots + \sqrt{m}U_m$ (certain of summand-operators can be zero). In turn, $U_k = \frac{1}{k} T P_k$, $k = 1, 2, \ldots, m$. Thus, $\mathfrak{M} = \mathfrak{M}_\varphi$ can be described as a C^*-algebra generated by the operator T and the subalgebra $\mathrm{M}(X)$.
To compute the conjugate operator T^* we introduce some notations. Let $E_y^n = \{x \in X : \varphi^n(x) = y\}$, $n = 0, 1, 2 \ldots$. We assume $E_y^0 = y$ and let the complete preimage of an element $y \in X$ be $E_y = E_y^1 = \{x \in X : \varphi(x) = y\}$. An element y for which $E_y = \emptyset$ we call *φ-initial*. Obviously, $E_{y_1}^n \cap E_{y_2}^n = \emptyset$ for $y_1 \neq y_2$, and then for each n the set X can be represented as a disjoint union of these subsets, $X = \bigcup_{y \in X} E_y^n$. Respectively, for any fixed positive integer n we have
$$l^2(X) = \bigoplus_{y \in X} l^2(E_y^n). \tag{2}$$

Now, evidently the conjugate operator T^* can be calculated by the formula

$$(T^*_\varphi f)(y) = \begin{cases} \sum\limits_{x \in E_y} f(x), & \text{if } E_y \neq \emptyset; \\ 0, & \text{if } E_y = \emptyset. \end{cases} \tag{3}$$

Accordingly, the set X can be represented as a disjoint union of the subsets $X_k = \{y \in X : \operatorname{card} E_y = k\}$, and then we obtain an orthogonal decomposition $l^2(X) = \oplus_{k=0}^m l^2(X_k)$ (assuming $l^2(X_k)$ as a $\{0\}$ space if X_k is empty) by the subspace described as $l^2(X_k) = \{f \in l^2(X) : T^*Tf = kf\}$. Then $T^*T = \oplus_{k=1}^m kP_k$, where P_k is the projection onto the subspace $l^2(X_k)$.
Similarly, $TT^* = \oplus_{k=1}^m kQ_k$, where Q_k is the projection onto the subspace $l_k^2 = \{f \in l^2(X) : TT^*f = kf\}$ for all $k \neq 0$. Defining l_0^2 as the orthogonal complement to all remaining l_k^2 we obtain $l^2(X) = \bigoplus_{k=0}^m l_k^2$.

Functions from the family $\{e_x, x \in X\}$ where $e_x(y) = \delta_x^y$ (Kronecker symbol) forms an orthonormal basis on the Hilbert space $l^2(X)$ and the subspaces $l^2(X_k)$ mentioned above. The family $\{g_y = \frac{1}{\sqrt{k}} \sum_{x \in E_y} e_x, y \in X_k\}$, forms an orthonormal basis in the space l_k^2 when $k \neq 0$. The operator T^* acts on the basis elements as $T^*e_x = e_{\varphi(x)}$, $x \in X$.

Projections P_k and Q_k are equivalent and mutually non permutable in general. The respective partial isometry U_k, $k \neq 0$, is defined as follows:

$$U_k e_y = \begin{cases} g_y & \text{if } y \in X_k \\ 0, & \text{if } y \notin X_k. \end{cases} \tag{4}$$

Accordingly,

$$U_k^* g_y = \begin{cases} e_y, & \text{if } y \in X_k \\ 0, & \text{if } y \notin X_k. \end{cases} \tag{5}$$

Obviously, the operator $U = U_1 + U_2 + \cdots + U_m$ is a partial isometry. If φ is surjective (resp. bijective), then U is an isometry (resp. unitary).

Remark 1 *In the case when φ is surjective the operator U generates an inner endomorphism β_U of the algebra $\mathfrak{M}$,*

$$\beta_U(A) = UAU^*$$

which is an automorphism if φ is a bijection. This endomorphism plays the central role in representing the algebra $\mathfrak{M}$ as a crossed product in Section 3.

We give important commutation relation between the generators (cf.[6]):

Proposition 1 ([6]) *For each function f from $B(X)$:*
(i) $TM_f = M_{T_f}T$
(ii) $T^*M_fT = M_{T^*f}$.

Remark 2 *Similar relations can be deduced for the partial isometries mentioned above: for each function f from $B(X)$ and every positive integer k*
(i) $U_k M_f = M_{Tf} U_k$
(ii) $U_k^* M_f U_k = \frac{1}{k} M_{(T^*Tf)I_k}$
where I_k is the indicator of the set X_k.

The algebra $\mathrm{M}(X) \subset \mathfrak{M}_\varphi$ contains all projections $\{P_Y := M_{I(Y)}, Y \subset X\}$, where $I(Y)$ is the indicator of the set Y, and particulary one-dimensional projections $P_x := P_{\{x\}}$. Thus, if $(f, e_x) \neq 0$ for a function f and a point x, then $e_x \in \mathfrak{M}f$. If the graph related to φ is connected, then the algebra $\mathfrak{M}$ is irreducible and contains the ideal $K(l^2(X))$ of compact operators.
Elements of the set $E(X) = M(X) \bigcup \mathcal{U} \bigcup \mathcal{U}^*$ we call *elementary monomials.* The notion of *index* (ind) can be defined for each operator from $A \in E(X)$, notably, $\mathrm{ind}(A) = 0\,(1, -1)$ for $A \in M(X)$ ($\mathcal{U}$, or $\mathcal{U}^*$, respectively). We assume the index of zero operator to be 0. Each finite product of elementary monomials we call *monomial*, denoting their set by $\mathrm{Mon}(X)$, and considering $\mathrm{ind}V$ for $V \in \mathrm{Mon}(X)$ as the sum of the indices of the factors. It was proved in [6] that the index of a monomial does not depend of its representation as a product of elementary monomials.
The *length* $\mathrm{d}(V)$ of a monomial V is the least number of partial isometries from $\mathcal{U} \bigcup \mathcal{U}^*$ participating in its representation as a product. Obviously, linear combinations of monomials are dense in $\mathfrak{M}$ and the set $\mathrm{Mon}(X)$ forms a semigroup with respect to multiplication operation.
It was shown in [6] that by using the notion of index a $\mathbb{Z}$-grading of $\mathfrak{M}$ can be established, namely, $\mathfrak{M} = \overline{\oplus_{n\in\mathbb{Z}} \mathfrak{M}_{\varphi,n}}$, where $\mathfrak{M}_n$ is a subspace generated by monomials of index n.

2 Action of the unit circle

First of all we recall basic facts which will be used in further. Let $\mathfrak{A}$ be a C^*-algebra and α be an action of the unit circle S^1 on $\mathfrak{A}$. For any $n \in \mathbb{Z}$ the *spectral subspace*

$$\mathfrak{A}_n = \{A \in \mathfrak{A} : \alpha_z(A) = z^n A \quad \text{for} \quad z \in S^1\}$$

and *spectral projection* $\mathcal{P}_n : \mathfrak{A} \longrightarrow \mathfrak{A}$,

$$\mathcal{P}_n(A) = \int\limits_{\mathrm{T}} z^{-n} \alpha_z(A)\, \mathrm{d}z$$

are determined. Obviously, the range of the projection $\mathcal{P}_n$ is the spectral subspace $\mathfrak{A}_n$. By the way, the subalgebra $\mathfrak{A}_0$ is the *fixed point subalgebra* under the mentioned action.

Theorem 1 *There exists a continuous morphism α of the group S^1 into the automorphism group Aut($\mathfrak{M}$) such that the corresponding n-th spectral subspace coincides with the subspace $\mathfrak{M}_n$.*

Proof. Define an action α of S^1 on the elements V of Mon(X) by the formula

$$\alpha_z(V) = z^{\mathrm{ind}V}\, V.$$

It is evident that $\mathfrak{M}_n$ is the n-th spectral subspace. □

Remark, that the *stationary subalgebra* $\mathfrak{M}_0$ is fixed point subalgebra under the action α, i.e. the grading is generated by the covariant system $(\mathfrak{M}, S^1, \alpha)$. Moreover, the mentioned action is semi-saturated which means that the algebra $\mathfrak{M}$, as a C^*-algebra is generated by the fixed point subalgebra and the first spectral subspace $\mathfrak{A}_1$ (see [1]). It is easy to verify that the mapping defined as

$$\mathcal{P}_0(A) = \int\limits_{\mathbf{T}} \alpha_z(A)\mathrm{d}z$$

is a conditional expectation onto the fixed point subalgebra. Obviously, if $A = \sum_{k=-n}^{m} A_k$, where $A_k \in \mathfrak{M}_k$, then of course, $\mathcal{P}_0(\sum_{k=-n}^{m} A_k) = A_0$.
Let us now turn to the study of the structure of the fixed point subalgebra. Denote by $\mathfrak{M}_0^{(n)}$ the C^*-algebra generated by monomials V with $\mathrm{ind}(V) = 0$ and $\mathrm{d}(V) \leq 2n$. There is a directed chain of C^*-algebras,

$$\mathfrak{M}_{\varphi,0}^{(1)} \subset \mathfrak{M}_{\varphi,0}^{(2)} \subset \cdots \subset \mathfrak{M}_{\varphi,0}^{(n)} \subset \cdots,$$

and

$$\mathfrak{M}_{\varphi,0} = \overline{\bigcup_{s=1}^{\infty} \mathfrak{M}_{\varphi,0}^{(n)}}.$$

It is easy to understand, that each $l^2(E_y^n)$ in the representation (2) is a finite-dimensional space which is invariant with respect to the monomials of zero index and the length $\mathrm{d} \leq 2n$ (see [6], Corollary 3.3), and consequently, with respect to all operators from the algebra $\mathfrak{M}_0^{(n)}$.
Let $I_{y,n}$ be the indicator of the set E_y^n. Then, the operator $P_{y,n} := M_{I_{y,n}}$ belongs to $\mathfrak{M}_0$ and is a projection onto the subspace $l^2(E_y^n)$. Obviously, the operators $\{P_{y,n}\}$ form in $\mathfrak{M}$ a block system such that all monomials from $\mathfrak{M}_0^{(n)}$ are block-diagonalized. Note that each zero index monomial from the algebra $\mathfrak{M}$ is block-diagonalized.

Lemma 1 *There exists on the algebra $\mathfrak{M}$ a conditional expectation onto the subalgebra of multipliers.*

Proof. Let us define $\mathcal{P}_M(A) = \oplus_{x\in X} P_{\{x\}} A P_{\{x\}}$ for $A \in \mathfrak{M}$. It can be checked at once that $\mathcal{P}_M$ is a projection of norm one hence

$$\mathcal{P}_M : \mathfrak{M} \longrightarrow \mathrm{M}(X)$$

is a conditional expectation.
□

The following interesting observation is presented for completness.

Corollary 1 *There exists a trace state on the algebra* $\mathfrak{M}$.

Proof. Remind that each state on a commutative C^*-algebra is a trace state. Thus, if τ is a state on $\mathfrak{M}$, then $\tau \circ \mathcal{P}_M$ is a trace state on $\mathfrak{M}$. □

3 Crossed product structure on $\mathfrak{M}$

The next goal is to show that in some cases the algebra $\mathfrak{M}$ can be represented as the crossed product in the sense of P.J. Stacey, [7]. We bring slightly simplified definitions formulated in terms of covariant representations.
For any C^*-algebra $\mathfrak{A}$ and a star-endomorphism α we use a standard notation $\mathfrak{A}_\infty$ for the inductive limit of the sequence

$$\mathfrak{A} \xrightarrow{\alpha} \mathfrak{A} \xrightarrow{\alpha} \mathfrak{A} \xrightarrow{\alpha} \mathfrak{A} \xrightarrow{\alpha} \cdots .$$

Let $\mathfrak{A}$ be a unital C^*-algebra and β be a $*$-endomorphism of $\mathfrak{A}$. The pair (π, V) is called the covariant representation of the system $(\mathfrak{A}, \beta)$ if π is a non-degenerated representation $\pi : \mathfrak{A} \longrightarrow B(H)$ and V is an isometry of $B(H)$ such that $\pi(\beta(a)) = V\pi(a)V^*$ for every $a \in \mathfrak{A}$ (that is $\pi \circ \beta = \beta_V \circ \pi$). The crossed product associated with a given system $(\mathfrak{A}, \beta)$ with $\mathfrak{A}_\infty \neq 0$ is a unital C^*-algebra $\mathfrak{B}$ together with an identity preserving $*$-homomorphism $\nu : \mathfrak{A} \longrightarrow \mathfrak{B}$ and an isometry u in $\mathfrak{B}$ such that

(i) $\nu(\beta(a)) = u\nu(a)u^*$ for all $a \in \mathfrak{A}$ (that is, $\nu \circ \beta = \beta_u \circ \nu$)

(ii) for every covariant representation (π, V) of the system $(\mathfrak{A}, \beta)$ there exists a non-degenerated representation τ of $\mathfrak{B}$ in H_π with $\tau \circ \nu = \pi$, and $\tau(u) = V$

(iii) the algebra $\mathfrak{B}$ is generated by elements of the form $\nu(a)u^n u^{*m}$.

Theorem 2 *Let φ be a surjective (non injective) mapping. Then the algebra $\mathfrak{M}_\varphi$ is the crossed product associated to the system $(\mathfrak{M}_{\varphi 0}, \beta_{U_\varphi})$ consisting of the fixed point subalgebra and the standard inner endomorphism.*

Proof. Under the conditions, the operator $U = U_\varphi$ from $\mathfrak{M} = \mathfrak{M}_\varphi$ (see remark 1) is isometric (non unitary). It is evident that the fixed point subalgebra is invariant under the action of $\beta = \beta_U$. Since $\beta^n(A) = U^n A U^{*n}$ and so $U^{*n}\beta^n(A)U^n = A$, we have $\|\beta^n(A)\| = \|A\|$. Then $\mathfrak{M}_{0\infty} \neq 0$ and the pair (id, U) is a covariant representation of the system $(\mathfrak{M}_0, \beta_U)$.
It remains to show that the algebra $\mathfrak{M}$ is generated by the fixed point subalgebra and the isometry U. Indeed, finite sums of operators $A_n \in \mathfrak{M}_n$ are dense in $\mathfrak{M}$ ([6]). In the case $n > 0$ we have $A_n = A_n U^{*n} U^n$ with $A_n U^{*n} \in \mathfrak{M}_0$. If $n < 0$, then similarly $A_n = U^{*-n} U^{-n} A_n$ with $U^{-n} A_n \in \mathfrak{M}_{\varphi 0}$. □

4 Example

Let φ be the right shift on $\mathbb{Z}_+$, $\varphi(n) = n+1$. The corresponding C^*-algebra $\mathfrak{M}_\varphi$ is denoted as usual by $\mathfrak{M}$. Thus, the algebra is generated by an isometric operator $W = T^*$ and by multipliers. According to the known theorem of Coburn, operator W generates the so called Toeplitz algebra $\mathfrak{T}$.
Our aim is to give a detailed description of the algebra $\mathfrak{M}$.

Lemma 2 *For each function $f \in B(X)$ there exist the functions f_1, f_2 from $B(X)$ such that $M_f W = W M_{f_1}$ and $W M_f = M_{f_2} W$.*

Proof. . The first relation follows immediately from the proposition 1, with $f_1 = f \circ \varphi$. As f_2 one can take the following function

$$f_2(n) = \begin{cases} f(n-1), & \text{if } n \neq 0 \\ 0, & \text{if } n = 0. \end{cases}$$

Equality can be checked by simple substitution. □

Corollary 2 *Evidently,* $\mathfrak{M} = \overline{M(\mathbb{Z}_+)\mathfrak{T}}$.

The $\mathbb{Z}$-grading of the algebra $\mathfrak{M}$ can be described more precisely.

Corollary 3 *We have* $\mathfrak{M} = \bigoplus_{n\in\mathbb{Z}} \overline{\mathfrak{M}_n}$ *where*

$$\mathfrak{M}_n = \begin{cases} \mathrm{M}(\mathbb{Z}_+)W^n, & \textit{if} \quad n > 0 \\ W^{*n}\mathrm{M}(\mathbb{Z}_+), & \textit{if} \quad n < 0. \end{cases}$$

Proof. Since the subalgebra $\mathfrak{T}_0$ corresponding to zero in the $\mathbb{Z}$-grading of the Toeplitz algebra is generated by the identity operator and by the one-dimensional basis projections, it is contained in $\mathrm{M}(\mathbb{Z}_+)$. □

The algebra $\mathfrak{M}$ contains the ideal of compact operators $K(l^2(\mathbb{Z}_+)$. Let us consider the quotient algebra $\widehat{\mathfrak{M}} = \mathfrak{M} \diagup K(l^2(\mathbb{Z}_+))$ (the quotient image always will be denoted by hat). Despite the fact that quotient of the Toeplitz algebra is commutative as well as the algebra of multipliers, algebra $\widehat{\mathfrak{M}}$ is not commutative. However, there is a commutative subalgebra $\widehat{\mathrm{M}(\mathbb{Z}_+)}$ in $\mathfrak{M}$. Moreover, this algebra is masa (as an image of masa in Calkin algebra), [8].
The following result shows that the quotient image of our algebra is a usual crossed product of the commutative subalgebra and the group $\mathbb{Z}$.

Lemma 3 *The algebra $\widehat{\mathfrak{M}}$ is the crossed product associated with the system $(\widehat{\mathrm{M}(\mathbb{Z}_+)}, \beta_{\widehat{W}})$, consisting of the quotient image of the algebra of multipliers and inner automorphism $\beta_{\widehat{W}}$.*

The algebra $\widehat{\mathfrak{M}}$ inherits the grading of the algebra $\mathfrak{M}$ (see 3)

$$\widehat{\mathfrak{M}} = \bigoplus_{n \in \mathbb{Z}} \overline{\widehat{\mathfrak{M}_n}}$$

where the subspaces $\widehat{\mathfrak{M}_n}$ can be described as

$$\widehat{\mathfrak{M}_n} = \begin{cases} \widehat{\mathrm{M}(Z_+)}\widehat{W}^n, & \text{if } n > 0 \\ \widehat{W}^{*|n|}\widehat{\mathrm{M}(\mathbb{Z}_+)}, & \text{if } n < 0 \end{cases}$$

where the operator $\widehat{W}$ is obviously unitary.

Corollary 4 *There exists a short exact sequence*

$$0 \longrightarrow K(l^2(\mathbb{Z}_+)) \longrightarrow \mathfrak{M} \longrightarrow \widehat{\mathfrak{M}} \longrightarrow 0.$$

References

[1] R. Exel, *Circle actions on C^*-algebras, partial automorphisms and generalized Pimsner-Voiculescu exact sequence*, J. Funct. Anal. **122** (2) (1994), 361–401.

[2] R. Exel, A. Vershik, *C^*-algebras of irreversible dynamical systems*, Canad. J. Math., **58** (1) (2006), 39–63.

[3] V. Arzumanian, *Operator algebras associated with non-singular endomorphisms of Lebesgue space*, J. Conterporary Mathematical Analysis, **21** (6) (1986), 596–616.

[4] S.A. Grigoryan, A.Yu Kuznetsova, *C^*-algebras Generated by Mappings* [in Russian], Math. Notes **87**(5) (2010), 663–671.

[5] S. Grigoryan, A. Kuznetsova, *On a class of nuclear C^*-algebras*, Operator Theory Summer, Proc. of 23rd Int. Conf. on Operator Theory (Timisoara, Romania, 2010), 39–50.

[6] A.Yu. Kuznetsova, E.V. Patrin, *One class of C^*-algebras generated by a family of partial isometries and multiplicators*, Russian Mathematics (Iz. VUZ) **56**(6) (2012), 37–47.

[7] P.J. Stacey, *Crossed product of C^*-algebras by *-endomorphisms*, J. Austral. Math. Soc., Series A, **54** (1993), 204–212.

[8] S.Shelan, J. Steprans, *Masas in the Calkin algebra without the continiuum hypothesis*, J. Applied Analysis **17** (2011), 69–89.

A.Yu. Kuznetsova
Institute of Physics
Kazan Federal University
18, Kremlevskaya str.
20008 Kazan, Russia
alla.kuznetsova@gmail.com

Ye.V. Patrin
Institute of Physics
Kazan Federal University
18, Kremlevskaya str.
20008 Kazan, Russia
evgeniipatrin@mail.ru

Please, cite to this paper as published in
Armen. J. Math., V. **7**, N. 1(2015), pp. 50–58