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# Consistent Systems of Finite Dimensional Distributions

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*To Professor Arshag Hajian in honor of his anniversary*

**Abstract.** In the paper we discuss the problem of description of random fields by means of systems of finite dimensional probability distributions.
We present from a unified point of view a survey of the various such systems associated with random fields.
Then general system of finite dimensional distributions with a suitable consistency condition is introduced. The properties, relations with other systems, as well as the problems of existence and uniqueness of the corresponding random fields are discussed.

*Key Words:* measures in infinite dimensional spaces, random field, specification, conditional distributions, statistical mechanics.
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# Introduction

In the theory of random fields the crucial role play various systems of finite dimensional probability distributions generated by a field. Meaningful statements of the theory can be formulated in terms of such systems and each of them has its own contribution to the study of the random field.
This is the reason why the problem of description of random fields with the help of systems of finite dimensional distributions is quite important.
The precise formulation of the inverse problem is: for a system of finite dimensional distributions find appropriate consistency conditions under which there exists a random field generating the system. The problem of uniqueness of the random field is closely related to the just mentioned.

Historically the first such system was the Kolmogorov system of (absolute) finite dimensional distributions which uniquely determines a random field.
Another important systems (specifications) was introduced by R. Dobrushin in connection with the problem of definition of Gibbs measures in infinite dimensional spaces, see [7]. Specification is a set of finite dimensional distributions indexed by infinite boundary conditions. In contrast to Kolmogorov system a specification in general does not determine the random field uniquely. In *Statistical mechanics* the last circumstance is usually interpreted as the presence of a phase transition.
There is a disharmony between the Dobrushin conditions: the existence condition (quasilocality) is imposed on whole specification while the uniqueness condition is specified for one-point distributions only.
In the works of the second author and S. Dachian (see [3], [4]) this weakness has been eliminated. A consistent system of one-point distributions was introduced (one-specification) and shown that this system uniquely determines a Dobrushin specification and inherits all its main properties.
Thus, in Dobrushin theory it is enough to impose the conditions on the one-specifications only.
In the paper of the second author and A. Dalalyan [6] a class of systems of finite dimensional probability distributions with finite boundary conditions was introduced and studied. As in Kolmogorov case this system uniquely determines a random field. Moreover, the finite dimensional distributions of the constructed random field have the explicit form in terms of the initial system.
The close problems of description of random fields were considered in the papers by H.-O. Georgii [13], R. Fernandez and G. Millard ([10], [11]), R. G. Flood and W. G. Sullivan, [12] et al.

The present note have two purposes: first we give the descriptions of mentioned systems from a new point of view and then offer a general system of finite dimensional distributions reduced to the listed ones (as subsystems) by the appropriate choice of boundary functions and consistency conditions.

# 1 Preliminaries

In this section we provide some of the concepts, notations and agreements necessary for the further discussions.

## 1.1 Configuration space

Let $S$ be a countable set (keep in mind $S = \mathbb{Z}^\nu$ though the lattice structure does not exploit).
For any $V \subset S$ we denote the set of all n-point subsets of $V$ by $\mathcal{F}_n(V)$,

$\mathcal{F}_0(V) = \emptyset$, and the set of all finite subsets of $V$ by $\mathcal{F}(V) = \cup_{n=0}^{\infty}\mathcal{F}_n(V)$.
When $V = S$, we write $\mathcal{F}_n, \mathcal{F}$ instead of $\mathcal{F}_n(S)$ and $\mathcal{F}(S)$.
For any $V \in \mathcal{F}$ let $|V|$ be the number of elements of $V$.

Let a finite set $X$ (which elements will be referred as spins) be fixed once for all. For a non-empty $V \subset S$ the set of all mappings $x : V \to X$ (*configurations* on $V$) will be denoted by $X^V$.
If the subset $V$ is finite ($|V| = n$) then a configuration $x \in X^V$ will be called *finite* (or more precisely, $n$-configuration).
Let $d(x)$ denotes the domain of $x$, and $c(x) = S \setminus d(x)$. Denote by $x_V$ the restriction of $x$ to some $V \subset d(x)$, that is $d(x_V) = V$ and the values of these configurations coincide on $V$, naimely, $x_V(s) = x(s)$, $s \in V$.
If $|V| = 1, V = \{s\}$, $s \in S$ we always write $s$ instead of $\{s\}$.
Obviously, for an $s \in S$, the set $X^s = X^{\{s\}}$ is a copy of $X$; in such case the configuration defined on $\{s\}$ with a value $x$ will be referred symply as $x$, if it not led to a misunderstanding.

The set $X^S$ is a topological space with the (Tikhonov) topology (assuming on $X$ the discret topology). The base of open sets constitute the cylinder sets $U_x$, corresponding to finite configurations $x$:

$$U_x = \{z \in X^S : z(s) = x(s) \text{ for } s \in d(x)\}.$$

Real-valued function $f$ on $X^S$ is called *local* if there exists $V \in \mathcal{F}$ such that $f(x) = f(y)$ whenever $x_V = y_V$, and *quasi-local* if

$$\lim_{V \uparrow S} \sup_{x_V = y_V} |f(x) - f(y)| = 0.$$

The notion of quasi-locality is equivalent to the notion of continuity of functions in the just described topology.

We denote by $\mathcal{X}(S) = \cup_{V \subset S} X^V$ the set of all configurations and by $\mathcal{X}_{\mathcal{F}}(S) = \cup_{V \in \mathcal{F}} X^V$ the set of finite configurations.

For any two functions $x_1, x_2 \in \mathcal{X}(S)$ with $d(x_1) \cap d(x_2) = \varnothing$ we denote by $x := x_1x_2$ their *concatenation*, that is the function with the domain $d(x) = d(x_1) \cup d(x_2)$ and such that $x_{d(x_i)} = x_i$, $i = 1, 2$.
For $x \in \mathcal{X}(S)$ with $d(x) = d_1 \cup d_2$ where $d_1$ and $d_2$ are disjoint, there is the representation $x$ as a concatenation $x = x_1x_2$, where $x_i = x_{d_i}$, $i = 1, 2$.

Let $V \in \mathcal{F}$. Any configuration $z$ with $V \subset c(z)$ is called a *boundary function (condition)* related to $V$. The set of all boundary functions related to a finite $V$ we denote by $\mathcal{B}_V$.

## 1.2 Probability distributions and random fields

For any $V \in \mathcal{F}$, we denote by $\mathcal{D}_V$ the set of all positive (finite dimensional) probability distributions on $X^V$, that is for any $q \in \mathcal{D}_V$ supposed

$$\sum_{u \in X^V} q(u) = 1, \ q > 0.$$

Sometimes we write $q_V = q \in \mathcal{D}_V$ to indicate the volume determining $X^V$.

The usually considered metrics by variation $\rho_V$ on the set $\mathcal{D}_V$ is defined by the formula

$$\rho_V(q_1, q_2) = \frac{1}{2} \sum_{u \in X^V} |q_1(u) - q_2(u)|. \tag{1}$$

By a *random field* we understand a probability measure on the Borel $\sigma$-algebra generated by the Tikhonov topology.
There are many systems of finite dimensional distributions (probability measures on the cylinder sets) induced by the random field by means of which the random field can be restored.
In constructing of random field models starting from a finite dimensional distribution system the following problems are essential:

(i) find conditions on the system under which there exists a random field inducing the system
(ii) find the conditions guarantying its uniqueness
(iii) give an appropriate representation for the distribution systems.

The systems mentioned in Introduction are characterized by the choice of classes of finite subsets, appropriate boundary functions, and consistency conditions.

## 1.3 Precifications

Let $F$ be a subset of $\mathcal{F}$, and let $\varphi$ be a map $\varphi : F \to \cup_{I \in F} \mathcal{B}_I$, such that the imagee of each set $V \in F$ is a subset of $\mathcal{B}_V$, $\varphi(V) \subset \mathcal{B}_V$. As a finite dimensional probability distribution system $Q(F, \varphi)$ we understand the aggregation

$$\{q^z \in D_V : V \in F, \, z \in \varphi(V)\}.$$

Such a system $Q(F, \varphi)$ we call *precification*[1], or more precisely $(F, \varphi)$-*precification.* The sets from $F$ as well as the corresponding boundary conditions will be called *acceptable.*
Obviously, there is a largest precification (with $F = \mathcal{F}$ and $\varphi(\Lambda) = \mathcal{B}_\Lambda$ for any $\Lambda \in F$) which will be called *total.*

[1]The term *precification* ($\approx$ pre+specification) is introduced for brevity in considering arbitrary systems of distributions without any consistency conditions.

Each random field induces various kinds of precifications.

We say that a precification is *prescribing* if there exists random fields which induce it, and *determining* if it is prescribing and the related random field is unique.

We are interested not only in the problems of existence and uniqueness of a random field with given system of finite dimensional distributions, but also in its convenient representation in reasonable terms. To understand better what we mean, let us recall the characterization of conditional probabilities of Markov chains by transition matrices, the Gibbs representation (in terms of potential) of specifications, spectral representation for correlation functions of stationary random processes and so on.

Let $P = \{P_s,\ s \in S\}$ be a system of probability distributions on $X^s$.
We say that the system $P$ is *compatible* with a precification $Q(F, \varphi)$ if for any finite configuration $x$ and for all pairs of points $s, t \in V = d(x)$ the following relation holds on $Q(F, \varphi)$

$$q^{x_s}(x_{V\setminus s})P_s(x_s) = q^{x_t}(x_{V\setminus t)})P_t(x_t), \tag{2}$$

i.e. the product of the form $q^{x_s}(x_{d(x)\setminus s)})P_s(x_s)$ does not depend on the choice of points in the domain of a finite configuration.

## 1.4 Cyclic functions

Now we introduce certain functions related to a precification which plays an important role in the further considerations.
Let $x, u, y, v \in \mathcal{X}(S)$ be four $n$-configurations which domains satisfy the condition $d(y) \neq d(x) = d(u) \neq d(v)$, and $z$ be a boundary condition related to $d(x) \cup d(y) \cup d(v)$. Consider the real function (referred as $\alpha$-functions) related to a given precification:

$$\alpha^z(x, y, u, v) = q^{zx}(y)q^{zy}(u)q^{zu}(v)q^{zv}(x) \tag{3}$$

which obviously is invariant under the cyclic permutations of arguments, i.e.

$$\alpha^z(x, y, u, v) = \alpha^z(y, u, v, x).$$

An $\alpha$-function $\alpha^z$ is called *symmetric* if it is invariant under a transposition of similar (with coinciding domains) arguments, i.e.

$$\alpha^z(x, y, u, v) = \alpha^z(x, v, u, y). \tag{4}$$

If $d(y) = d(v)$ we call such function *n-cyclic* related to the considering precification. The function $\alpha^z$ is called *cyclic* if it is $n$-cyclic for any positive integer $n$.
We shall see later that symmetric functions appear as consistency conditions on precifications and play a crucial role in the problem of description of random fields.

### 1.5 Gibbs precification

Recall that a Gibbs measure in a finite volume (limited to some $\Lambda \in \mathcal{F}$) is given by the distribution (for simplicity we ignore the dependence on physical parameters)

$$q_\Lambda(x) = \Xi^{-1} \exp(H_\Lambda(x)),$$

where $x \in X^\Lambda$, $H_\Lambda(x)$ is a *Hamiltonian* (that is the *energy* of the system, assumed to have an additive form with respect to a *potential* $\Phi$ of interactions (for details see 2.3), and the *partition function* (normalizing factor)

$$\Xi = \sum_{u \in X^\Lambda} \exp(H_\Lambda(u)).$$

Obviously, the set of all such finite dimensional distributions is a precification without boundary conditions and all finite subsets acceptable.

Particularly, in the case of pair potential the Hamiltonian has the form

$$H_\Lambda(x) = \sum_{s,t \in \Lambda} \Phi(x_t, x_s).$$

Usually, the definition of a limiting Gibbs measure in an infinite volume requires the involvement of so called *thermodynamic limit* (when $\Lambda \uparrow S$ in some sense).

## 2 Systems of probability distributions

In this section we discuss various classes of precifications with special consistency conditions.

### 2.1 Kolmogorov systems

Classical Kolmogorov system can be described as a precification corresponding to the system of all finite subsets without boundary conditions. Thus, $F = \mathcal{F}$, and $\varphi(\Lambda) = \emptyset$, for all finite $\Lambda$.
The consistency condition is the following

$$(p_\Lambda)_I = p_I, \tag{5}$$

where $I \subset \Lambda$, $p_V = q_V \in D_V$ and $(p_\Lambda)_I$ here and henceforth denotes

$$(p_\Lambda)_I(x) = \sum_{u \in X^{\Lambda \setminus I}} p_\Lambda(xu).$$

Being too general, this system does not have a convenient representation and therefore has largely the theoretical significance.

A classical result claims that Kolmogorov system is a determining precification (see e.g. [15]).

Using Kolmogorov system one can introduce the notion of conditional distribution of a random field (see [7]). Let $V \in \mathcal{F}$ and $\Lambda_n$ be an increasing sequence of finite subsets, $V \cap \Lambda_i = \emptyset$, $i = 1, 2, \ldots$ with $\cup_i \Lambda_i = S \setminus V$. Assume a boundary condition $z$ is related to the set $S \setminus V$, and let $z_n = z_{\Lambda_n}$. Consider the following sequence of ratios (forming a martingal)

$$q^{z_n}(x) = \frac{p_{V \cup \Lambda_n}(x z_n)}{p_{\Lambda_n}(z_n)}, n = 1, 2, \ldots, x \in X^V. \tag{6}$$

It is known that a.s. there exists the limit

$$q^z(x) = \lim_{n \to \infty} q^{z_n}(x)$$

which is called the *conditional distribution* of the random field (by Dobrushin).

Now we return to the construction of random fields associated with precifications suggested in the previous section.

**Proposition 1** *Let $Q = Q(F, \varphi)$ be a precification with $F = \mathcal{F}$ and boundary conditions $\mathcal{B}_\Lambda$ related to any $\Lambda \in F$. Then the precification $Q$ is prescribing, that is there exists a random field constructed by means of $Q$.*

***Proof.*** Let $\Lambda_n, n \in \mathbb{N}$, be an increasing sequence of finite subsets of $S$ such that $\cup_{n \in \mathbb{N}} \Lambda_n = S$ and $z_n$ be a sequence of boundary functions from $\mathcal{B}_{\Lambda_n}$. For all $I \in \mathcal{F}$, by the diagonal method (applying the finiteness of $X$) one can chose a subsequence (for simplicity we do not change the notations) for which the following limit exists

$$\lim_{n \to \infty} (q_{\Lambda_n}^{z_n})_I(x) = p_I(x).$$

It is easy to verify that the system $\{p_I\}$ satisfies the Kolmogorov consistency condition, and then a random field is uniquely determined. $\square$

Particularly, a limiting Gibbs measure on $X^S$ can be obtained by the construction mentioned in Proposition 1. However, there is no an explicit relationship bewwen the constructed Gibbs random field and the initial precification. Nevertheless in *Statistical mechanics* a powerful method of correlation functions is developed which is devoid the mentioned defect and allows the complete investigation of the properties of the Gibbs field (see e.g. [18]).
Through the efforts of Dobrushin, Lanford and Ruelle (see [17]) the difficulties in main part managed to get around and the problem was reduced to the investigation of finite dimensional distribution systems.

## 2.2 Dobrushin systems (specifications)

Dobrushin systems have been proposed primarily to construct the more general Gibbs measures with given Hamiltonian in infinite volumes.

We present here the systems (specifications) of Dobrushin. The Dobrushin approach to the problems of existence and uniqueness of a random field with given specification can be found in the notable survey [17], and in the works of Dobrushin cited there (see also the complete collection of Dobrushin's papers at the site [20]).

We consider a Dobrushin system $\mathcal{D}$ as a precification with $F = \mathcal{F}$ and $\varphi(\Lambda) = \{z \in \mathcal{B}_\Lambda : c(z) = \Lambda\}$ for any $\Lambda \in F$ endowed with the following consistency relation (*Dobrushin condition*)

$$q^z(xy) = q^{zx}(y) \sum_{u \in X^J} q^z(xu), \tag{7}$$

for $x \in X^I$, $y \in X^J$, $I \cap J = \emptyset, I, J \in F_c$, and any $z$ with $c(z) = I \cup J$.
The term used now for a such system is *specification*.
Each specification $\mathcal{D}$ contains a subsystem $\mathcal{D}_n$ (called n-point subspecification) which elements are defined only on $X^\Lambda$ with $|\Lambda| = n$, $n \in \mathbb{N}$.

The following result shows when a specification is prescribing.

**Theorem 1 (Dobrushin, [7])** *Let $\mathcal{D}$ be a quasi-local specification. Then there exists a random field which conditional probability distribution system a.s. coincides with $\mathcal{D}$.*

Dobrushin also gave a condition under which the system is determining.

**Theorem 2 (Dobrushin, [9])** *Let $\mathcal{D}$ be a quasi-local specification.The related random field to be unique the following condition on one-point quasi-local specification is sufficient*

$$\sup_{s \in S} \sum_{t \in S \setminus s} \sup_{u,v} \rho_{t,s}(q^u, q^v) < 1, \tag{8}$$

*where the second* sup *is taken over all $u, v$ with $d(u) = d(v) = S \setminus s$, and $u(r) = v(r)$, $r \neq s, t$.*

## 2.3 Gibbs systems (specifications)

We introduce here the notion of Gibbs specification essential for the Dobrushin definition of a Gibbs random field with a given potential.

A real function $\Phi$ on configurations $\mathcal{X}(S)$ is called interaction *potential*.
Potential $\Phi$ is called *convergent* if the series

$$\sum_{J\in\mathcal{F}(S\setminus s)} \Phi(xz_J)$$

converges for all $s \in S$, $x \in X^s$, $z \in X^{S\setminus s}$. A potential $\Phi$ is called *uniformly convergent* if the convergence is uniform with respect to $z$.
For any convergent potential a specification $\mathcal{G} = \mathcal{G}(\Phi)$ can be constructed using the following Gibbs formulae for $x \in X^\Lambda$, $\Lambda \in \mathcal{F}$, $z \in X^{S\setminus\Lambda}$

$$q^z(x) = \frac{\exp(H^z(x))}{\sum\limits_{u\in X^\Lambda} \exp(H^z(u))}, \tag{9}$$

where

$$H^z(y) = \sum_{\emptyset\neq I\subset\Lambda}\ \sum_{J\subset\mathcal{F}(S\setminus\Lambda)} \Phi(y_I z_J). \tag{10}$$

Such a specification is called *Gibbsian* (with potential $\Phi$).
Remark, that when some $z \in X^S$ is fixed the elements of Gibssian specifications $\{q_\Lambda^{z_{S\setminus\Lambda}}\}$ determine a Gibbs precification (see 1.5), and then a limiting Gibbs random field with given potential may be constructed.
According to Dobrushin, a random field is called *Gibbsian with given potential* if its conditional distributions a.s. coincide with the Gibbs specification with the same potential.
Note that a set of Gibbs random fields with given potential is a simplex in an appropriate Banach space of random fields ([9]), and the set of limiting random fields coincides with the set of extreme points of this simplex. When a Gibbse random field with given potential is unique, it coincides with the limiting Gibbse random field with the same potential.

A consequence of Theorem 1 claims that the uniform convergence of potential implies the quasi-locality of the Gibbs specification and hence the existing of a Gibbs random field. It may be added that the uniqueness condition (Theorem 2) can be reformulated in terms of potential too.
There are a lot of works devoted to Gibbs representations of specifications, see e.g. [2], [16], [19].
Note that according to a theorem of Kozlov, [16], any quasi-local (Dobrushin) specification is Gibbsian.

## 2.4 1-specifications

As noted in the introduction, there is a discrepancy between the conditions of existence and uniqueness formulated by Dobrushin, the first of which is

imposed on the entire specification, while the second only to the one-point subsystem. R. Dobrushin himself in [7] and [9] posed the problem to deduce both results from the conditions on one-point subspecification only.
This problem was solved many years later in the papers [3], [4].
Now we present the main results of these works in our terms.
Consider a precification corresponding to the parameters $F_1 = \mathcal{F}_1$, and $\varphi_1(\Lambda) = \{z \in \mathcal{B}_\Lambda, c(z) = \Lambda\}, \Lambda \in F$ in assuming all cyclic functions $\alpha^z$ defined in (3) are symmetric.
Thus, as it was in the original form, the consistency condition is

$$q^{zx}(y)q^{zy}(u)q^{zu}(v)q^{zv}(x) = q^{zx}(v)q^{zv}(u)q^{zu}(y)q^{zy}(x) \tag{11}$$

for all functions $x, u, y, v$ such that $d(x) = d(u)$, $d(y) = d(v) \in F_1$, and $z$ is a boundary function from $\varphi_1(d(x) \cup d(y))$.
The system $\mathcal{R}_1$ with just described precification and related symmetric cyclic functions concerned as a consistency condition is called *1-specification.*

**Theorem 3 (Dashyan-Nahapetian, [3], [4] )** *Any one-point subspecification $\mathcal{D}_1$ of a specification $\mathcal{D}$ is a 1-specification. Conversely, each 1-specification is the one-point subspecification of a unique specification.*

Remark that as a consequence of this theorem we obtain that the quasi-locality of the one-point subspecification implies the quasi-locality of entire specification.
Another immediate corollary is that any 1-specification is prescribing.

**Definition 1** *A random field $\mathbb{P}$ is called* Gibbs random field *if*
*(i) $\mathbb{P}_\Lambda(y) > 0$ for any $\Lambda \in \mathcal{F}$ and $y \in X^\Lambda$*
*(ii) the limits*

$$q^z(x) = \lim_{\Lambda \uparrow S \setminus s} \frac{\mathbb{P}_{s\cup\Lambda}(xz_\Lambda)}{\mathbb{P}_\Lambda(z_\Lambda)}, \ s \in S, x \in X^s, z \in X^{S\setminus s}$$

*exist, are strictly positive, and the convergence is uniform with respect to $z$.*

If $\mathbb{P}$ is a Gibbs random field, then the system $\{q^z\}$ is a 1-specification and then by Theorem 3 a *canonical* specification can be restored uniquely.

**Theorem 4 ([5])** *The canonical specification of a Gibbs random field is Gibbsian with some potential. Conversely, any random field related to a determining Gibssian specification is Gibssian.*

This fact shows that an inner definition (that is without involvement of the notion of potential) of Gibbs random field can be given.
At the same time it gives a useful representation for conditional distributions of Gibbs random fields.

Remark, at the end, that one can consider the more general systems of $n$-point specifications $\mathcal{R}_n$ with the consistency conditions in terms of the cyclic functions, which have a certain theoretical interest. We reserve these questions for the further studies.

### 2.5 One-point finite conditional distribution systems

In the paper [6] the systems of one-point probability distributions and boundary functions with finite domains were considered.
Namely, the system $\mathcal{L}$ corresponds to the precification with $F_1 = \mathcal{F}_1(S)$, boundary conditions related to a set $\Lambda \in F_1$ as $\{z \in \mathcal{B}_\Lambda : |d(z)| < \infty\}$, and as a consistency condition the following relation assumed for any functions $x, y$ with $|d(x)| = |d(y)| = 1, d(x) \cap d(y) = \emptyset$, and boundary functions $z$ with $d(z) \in \mathcal{F}$, $d(x) \cup d(y) \subset c(z)$

$$q^z(x)q^{zx}(y) = q^z(y)q^{zy}(x). \tag{12}$$

The main result is the following.

**Theorem 5 (Dalalyan, Nahapetian)** *The system $\mathcal{L}$ is determining if and only if all related 1-cyclic functions with one-point boundary conditions are symmetric.*

Note that the finite dimensional distributions of the related random field has an explicit form (a representation).
Let $\Lambda = \{s_1, s_2, \ldots, s_n\}$, $n \in \mathbb{N}$. Then

$$P_\Lambda(x) = P_{s_1}(x_{s_1})q_{s_2}^{x_{s_1}}(x_{s_2}) \cdots q_{s_n}^{x_{s_1}x_{s_2}\cdots x_{s_{n-1}}}(x_{s_n}) \tag{13}$$

where $x \in X^\Lambda$ and

$$P_s(x) = \frac{q^y(x)}{q^x(y)} \left( \sum_{u \in X^s} \frac{q^y(u)}{q^u(y)} \right)^{-1}, \; x \in X^s, \, y \in X^t. \tag{14}$$

### 2.6 Palm systems

In this section we present a variant of a system of distributions which in particular cases associated with so called called Palm processes.

Fix $b \in \mathcal{X}(S)$, such that $B = d(b) \neq S$. As a set $F_B$ of acceptable subsets we take the set of all $V \in \mathcal{F}$, such that $V \cap B = \emptyset$.
For any $V \in F_B$ as acceptable boundary conditions the functions from $z \in \mathcal{B}_V$ such that $z_B = b$.
This precification with the consistency condition

$$q^b(xy) = q^b(x)q^{bx}(y), \, x \in X^I, y \in X^J, I \cup J \subset S \setminus B. \tag{15}$$

is called *Palm system* and is denoted by $\mathcal{P}_b$.

**Theorem 6** *Each Palm system uniquely determines a random field.*

***Proof.*** It is obvious that the system $\{q^b_\Lambda, \Lambda \in F_B\}$ is consistent in the Kolmogorov sense and then being a determining system on $S \setminus B$ is related to a unique random field $\mathbb{P}_b$ on $X^{S\setminus B}$.
Our immediate goal is to extent this measure on the entire $X^S$.
At first define a measure $\mathbb{P}_b$ on $X^B$:

$$\mathbb{P}_b(x) = \begin{cases} 1, & \text{if} \quad x = b \\ 0, & \text{if} \quad x \neq b. \end{cases}$$

The measure $\mathbb{P}_b$ is already defined on $X^{S\setminus B}$. It remains the case when configurations are taken from $X^V$ where for $V \cap B \neq \emptyset$.
Represent such a configuration $x$ as a concatenation of two configurations, $x = uv$, where $d(v) = V \setminus B$, and $d(u) = B \cap V$. Then we define

$$\mathbb{P}_b(x) = \mathbb{P}_b(uv) = \begin{cases} q^b(v), & \text{if} \quad u = b_{B\cap V} \\ 0, & \text{if} \quad u \neq b_{B\cap V}. \end{cases}$$

Verify that it is a probability distribution:

$$\sum_{x\in V} \mathbb{P}_b(x) = \sum_{v\in V\setminus B} \sum_{u\in B\cap V} \mathbb{P}_b(vu) = \sum_{v\in V\setminus B} \mathbb{P}_b(v b_{B\cap V}) = \sum_{v\in V\setminus B} q^b(v) = 1.$$

Verify now that the obtained system is consistent by Kolmogorov.
Let $I \subset V, x \in X^I$. Then

$$\sum_{y\in V\setminus I} \mathbb{P}_b(xy) = \sum_{v\in X^{V\setminus(B\cup I)}} \sum_{u\in X^{(V\setminus I)\cap B}} \mathbb{P}_b(x_{I\cap B} x_{I\setminus B} uv) =$$

$$\sum_{v\in X^{V\setminus(B\cup I)}} \mathbb{P}_b(b_{I\cap B} b(V \setminus I) \cap B x_{I\setminus B} v) = \sum_{v\in X^{V\setminus(B\cup I)}} \mathbb{P}_b(b_{V\cap B} x_{I\setminus B} v) =$$

$$\sum_{v\in X^{V\setminus(B\cup I)}} q^b(x_{I\setminus B} v) = q^b(x_{I\setminus B}) = \mathbb{P}_b(b_{I\cap B} x_{I\setminus B}) = \mathbb{P}_b(x).$$

□

In the theory of stochastic processes such a random processe in the case when $|B| = 1$ are called *Palm process* (see e.g. [14], [1]), and in the case, when $|B| = n, n > 1$ is called *Palm process of order n*.

Now we present Palm systems of a special kind which was used by Dobrushin in his construction of limiting Gibbs random fields.
Let $b \in \mathcal{X}(S)$ be a configuration such that $\Lambda = c(b) \in \mathcal{F}$, and let $q^b = q^b_\Lambda$ be a probability distribution on $X^\Lambda$. For any $I \subset \Lambda$ define the distribution $q^b_I$ on $X^I$ as $q^b_I(x) = \sum_{u\in\Lambda\setminus I} q^b(xu)$.
It is easy to verify that this system satisfies the condition (6) and then by Theorem 6 uniquely determines a random field which is called by Dobrushin a random field in the finite volume with fixed boundary condition.
This kind of random fields was applied for the construction of limiting Gibbs random fields (see [8]).

# 3 General systems

Now we introduce a general system $\mathcal{G}$, corresponding to the total precification with the following consistency condition

$$q^z(xy) = q^z(x)q^{zx}(y), \tag{16}$$

where $x, y$ are any two finite configurations, $d(x) \cap d(y) = \emptyset$, and any boundary function $z$ related to $d(x) \cup d(y)$ .

Let us show that the finite dimensional systems considered in the previous section are sub-precifications of the general one.

At first we note that if the empty boundary conditions are acceptable, then the system contains the Kolmogorov system $\{p_\Lambda = q_\Lambda, \Lambda \in \mathcal{F}\}$.
Indeed, in this case we have for $I \subset \Lambda$ and for any $x, y$ with $d(x) = I$, and $d(y) = \Lambda \setminus I$

$$p(xy) = p(x)q^x(y),$$

and summarizing both sides of this equality over all $y \in X^{\Lambda \setminus I}$ we obtain the Kolmogorov consistency condition (5) and then the existence and uniqueness of a random field. Remark also that all another elements of the system are generated by the field.

Taking into account the mentioned case *we assume $\varphi$ is such that $d(z) \neq \emptyset$ in what follows.*

Consider now Dobrushin specifications.

**Proposition 2** *Suppose a subsystem $\mathcal{E}$ of $\mathcal{G}$ is specified only by the choice of boundary conditions $z$ being such that $c(z) = \Lambda$ for $\Lambda \in \mathcal{F}$. Then the system $\mathcal{E}$ is a Dobrushin specification.*

***Proof.*** It is sufficient to show that the consistency condition (7) follows from the condition (16).
Indeed,

$$\sum_{y \in X^J} q^z(xy) = \sum_{y \in X^J} q^z(x)q^{zx}(y) = q^z(x) \sum_{y \in X^J} q^{zx}(y) = q^z(x).$$

Thus,

$$q^z(xy) = q^z(x)q^{zx}(y) = q^{zx}(y) \sum_{u \in X^J} q^z(xu).$$

□

Taking into account Theorem 3 we obtain as a corollary that any 1-specifications $\mathcal{R}_1$ is a subsistem of $\mathcal{G}$ too.

Evidently, any one-point conditional distribution system $\mathcal{L}$ (which by definition has deal with only the boundary conditions with finite domains) is also a subsystem of $\mathcal{G}$.
At last it is easy to understand that the Palm systems also are subsystems of a general one.

We return to the general systems.
Let us show now that the main property of cyclic functions (symmetricity) related to the system $\mathcal{G}$ is satisfied automatically.

**Proposition 3** *All functions $\alpha^z$ defined in Subsection 3 by the elements of a general system $\mathcal{G}$ are symmetric.*

***Proof.*** We obtain from the relation (16) in the notations of Subsection 3

$$\alpha^z(x,y,u,v) = q^{zx}(y)q^{zy}(u)q^{zu}(v)q^{zv}(x) = \frac{q^z(xy)q^z(yu)q^z(uv)q^z(vx)}{q^z(x)q^z(y)q^z(u)q^z(v)} =$$

$$\frac{q^z(xv)q^z(vu)q^z(uy)q^z(yx)}{q^z(x)q^z(v)q^z(u)q^z(y)} = q^{zx}(v)q^{zv}(u)q^{zu}(y)q^{zy}(x) = \alpha^z(x,v,u,y).$$

□

**Proposition 4** *Let $P = \{P_s, s \in S\}$ be a system of probability distributions on $X$ compatible with the system $\mathcal{G}$. Then the system of finite dimensional distributions $\mathcal{P} = \{p_\Lambda, \Lambda \in \mathcal{F}\}$ defined as*

$$p_\Lambda(x) = q^{x_s}(x_{\Lambda\setminus s})P_s(x(s)), \;\; s \in \Lambda$$

*is a determining system consistent in Kolmogorov sense.*

***Proof.*** We have for any $s \in I$

$$(p_\Lambda)_I(x) = \sum_{u\in X^{\Lambda\setminus I}} p_\Lambda(xu) = \sum_{u\in X^{\Lambda\setminus I}} p_\Lambda(x_s x_{I\setminus s} u) = \sum_{u\in X^{\Lambda\setminus I}} q^{x_s}(x_{I\setminus s}u)P_s(x(s)).$$

Applying 16 we obtain

$$(p_\Lambda)_I(x) = \sum_{u\in X^{\Lambda\setminus I}} q^{x_s}(x_{I\setminus s})q^{x_I}_{\Lambda\setminus t}(u)P_s(x(s)) = q^{x_s}(x_{I\setminus s})P_s(x(s)) = p_I(x).$$

□

In some cases the existence of compatible systems can be guaranteed.
We formulate this fact as a theorem, the proof of which in substantial part is borrowed from the book by the second author and S. Dachian, preparing for publication.

**Theorem 7** *Let $\mathcal{G}$ be a general system such that all 1-cyclic functions with one-point boundary conditions are symmetric. Then the system of distributions $P = \{P_s,\, s \in S\}$ with*

$$P_s(x) = \frac{q^y(x)}{q^x(y)} \left( \sum_{u \in X^s} \frac{q^y(u)}{q^u(y)} \right)^{-1}, \; x \in X^s,\, y \in X^t \tag{17}$$

*is compatible with respect to $\mathcal{G}$, that is the system $\mathcal{G}$ is determining.*

***Proof.*** Firstly we have to show the definition (17) is correct, that is does not depend on the choice of a configuration $y$. We have to show that for any $v \in X^r, r \in S$

$$\frac{q^v(x)}{q^x(v)} \left( \sum_{u \in X^s} \frac{q^v(u)}{q^u(v)} \right)^{-1} = \frac{q^y(x)}{q^x(y)} \left( \sum_{u \in X^s} \frac{q^y(u)}{q^u(y)} \right)^{-1}. \tag{18}$$

Let us show for any quadruple $x, y, u, v$, where $x, u \in X^s, y \in X^t, v \in X^r$ that the functions $\alpha$ are symmetric $\alpha(v, x, y, u) = \alpha(v, u, y, x)$, that is

$$q^v(x^s) q^{x^s}(y) q^y(u) q^u(v) = q^v(u) q^u(y) q^y(x^s) q^{x^s}(v) \tag{19}$$

If $t = s$ this relation means that $\alpha$ is a 1-cyclic symmetric function, which is the requirement of the theorem.
If the contrary, $t \neq s$, then we have using (16)

$$q^v(x)q^x(y)q^y(u)q^u(v) = \frac{q^v(xy)}{q^{vx}(y)} \cdot \frac{q^x(yv)}{q^{xy}(v)} \cdot \frac{q^y(uv)}{q^{yu}(v)} \cdot \frac{q^u(vy)}{q^{uv}(y)} =$$

$$\frac{q^v(uy)}{q^{vu}(y)} \cdot \frac{q^u(yv)}{q^{uy}(v)} \cdot \frac{q^y(xv)}{q^{yx}(v)} \cdot \frac{q^x(vy)}{q^{xv}(y)} = q^v(u)q^u(y)q^y(x)q^x(v)$$

since

$$\frac{q^v(xy)}{q^v(uy)} = \frac{q^{vy}(x)q^v(y)}{q^{vy}(u)q^v(y)} = \frac{q^y(xv)}{q^y(uv)}.$$

By summing over both parts of the equality 19 with respect to $u$

$$\sum_{u \in X^s} \frac{q^v(x)q^y(u)}{q^x(v)q^u(y)} = \sum_{u \in X} \frac{q^y(x)q^v(u)}{q^x(y)q^u(v)}$$

we obtain the relation equivalent to (18) which means that the system $P$ is well defined.
Now we show that this system is compatible with the system $\mathcal{G}$.
Let $x$ be a finite configuration with the domain $d(x) = \Lambda = V \cup \{s\} \cup \{t\}$.
At first let us verify the compatibility condition (2) for the case $V = \emptyset$ i.e. for $\Lambda = \{s, t\}$:

$$q^{x_s}(x_t)P_s(x_s) = q^{x_t}(x_s)P_t(x_t). \tag{20}$$

In other terms we have to prove for any $x \in X^s$ and $y \in X^t$ the relation

$$q^x(y)P_s(x) = q^y(x)P_t(y) \tag{21}$$

Using the relation (19) which establishes the symmetricity of the functions $\alpha$ we obtain for $x, u \in X^s$ and $y, v \in X^t$

$$\frac{q^y(x)q^x(v)q^v(u)}{q^v(x)} = \frac{q^x(y)q^y(u)q^u(v)}{q^u(y)}$$

Using this equality, we get

$$q^y(x)\sum_{v\in X^t}\frac{q^x(v)}{q^v(x)} = \sum_{v\in X^t}\frac{q^y(x)q^x(v)}{q^v(x)} = \sum_{u\in X^s}\sum_{v\in X^t}\frac{q^y(x)q^x(v)q^v(u)}{q^v(x)} =$$

$$\sum_{u\in X^s}\sum_{v\in X^t}\frac{q^x(y)q^y(u)q^u(v)}{q^u(y)} = \sum_{u\in X^s}\frac{q^x(y)q^y(u)}{q^u(y)} = q^x(y)\sum_{u\in X^s}\frac{q^y(u)}{q^u(y)}$$

Then

$$q^y(x)\left(\sum_{u\in X^s}\frac{q^y(u)}{q^u(y)}\right)^{-1} = q^x(y)\left(\sum_{v\in X^t}\frac{q^x(v)}{q^v(x)}\right)^{-1},$$

and at last

$$q^x(y)P_s(x) = q^x(y)\frac{q^y(x)}{q^x(y)}\left(\sum_{u\in X^s}\frac{q^y(u)}{q^u(y)}\right)^{-1} =$$

$$q^y(x)\frac{q^x(y)}{q^y(x)}\left(\sum_{v\in X^t}\frac{q^x(v)}{q^v(x)}\right)^{-1} = q^y(x)P_t(y).$$

Now, assume that $|\Lambda| > 2$.
Combining 20 with the general consistency condition (16) we obtain

$$q^{x_s}(x_{\Lambda\setminus s})P_s(x_s) = q^{x_s}(x_V x_t)P_s(x_s) = q^{x_s x_t}(x_V)q^{x_s}(x_t)P_s(x_s) =$$

$$q^{x_s x_t}(x_V)q^{x_t}(x_s)P_t(x_t) = q^{x_t}(x_V x_s)P_t(x_t) = q^{x_t}(x_{\Lambda\setminus t)})P_t(x_t).$$

□

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