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Black-Scholes Formula for Asian Option with Several Futures

M. E. Alaei

Abstract. The paper suggests the Black-Scholes type formulas for some options. The Asian options for several futures with depending components and uniform time steps are investigated.

Key Words: Black-Scholes formula, Asian option for several futures
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Introduction

Let $S(t)$ be a commodity price at a moment t. This price is a random variable and generates some risks. To decrease of risks the insurance companies can pose the future contracts of the following type: Today ($t = 0$) a businessman pays to an insurance company F dollars. If at a fixed time moment $t = T$ will be $S(T) > K$, where K is a fixed constant, then the insurance company will pay to the businessman $S(T) - K$ dollars. If $S(T) < K$, then there are no payments. Then $F - E[S(T) - K]_+$ is the expectation of winning of the insurance company, where

$$X_+ = \begin{cases} X, & \text{if } X \geq 0, \\ 0 & \text{if } X \leq 0. \end{cases}$$

Also the expectation of winning of the businessman is equal to $E[S(T) - K]_+ - F$. To make this contract equitable, this expression should be zero.

The aim of the present paper is to calculate the expectation $E[S(T) - K]_+$ for Asian option with several futures.

1 Auxiliary Lemmas

For calculation of expectation $E[S(T) - K]_+$ we need some lemmas. Let ξ be a normal random variable with mean a and variance σ^2. We use the notation $\xi \sim N(a, \sigma^2)$. Let $f_{a,\sigma}(x)$ be the density function of ξ

$$f_{a,\sigma}(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-a)^2}{2\sigma^2}\right],$$

while Φ be cumulative normal distribution function with zero mean and unit variance.

Lemma 1 *For any number B the following identity holds*

$$\int_B^\infty e^x f_{a,\sigma}(x)dx = \exp\left(a + \frac{\sigma^2}{2}\right)\Phi\left(\frac{a+\sigma^2-B}{\sigma}\right).$$

Proof. It is well-known that $\Phi(B) = 1 - \Phi(-B)$ and

$$\int_B^\infty f_{a,\sigma}(x)dx = \Phi\left(\frac{a-B}{\sigma}\right) \tag{1}$$

Using the change of variables $z = \frac{x-a}{\sigma}$, we get

$$I = \int_B^\infty e^x f_{a,\sigma}(x)dx = e^a \int_{(B-a)/\sigma}^\infty \frac{e^{-z^2/2+\sigma z}}{\sqrt{2\pi}}dz =$$

$$e^a \int_{(B-a)/\sigma}^\infty \frac{e^{-[(z-\sigma)^2-\sigma^2]/2}}{\sqrt{2\pi}}dz.$$

Substituting $z = t + \sigma$ in the last integral and using (2.1) we obtain

$$I = e^{a+\sigma^2/2}\Phi(\sigma - (B-a)/\sigma).$$

□

Lemma 2 *([4], page 266.). Let $\xi \sim N(a, \sigma^2)$. For any positive number K the following equality holds*

$$E(e^\xi - K)_+ = \exp\left(a + \frac{\sigma^2}{2}\right)\Phi\left(\frac{a+\sigma^2-\ln K}{\sigma}\right) - K\Phi\left(\frac{a-\ln K}{\sigma}\right). \tag{2}$$

Proof. If $x < \ln K$, then $(e^x - K)_+ = 0$. Therefore

$$E(e^\xi - K)_+ = \int_{\ln K}^\infty e^x f_{a,\sigma}(x)dx - K\int_{\ln K}^\infty f_{a,\sigma}(x)dx.$$

It remains to apply Lemma 1. □

From Lemma 2 we can obtain so called "Black-Scholes formula" for single option.

Theorem 1 *([1]). Let $S(t)$ be a commodity price, such that $\ln S(t)$ has normal distribution with mean $\ln S(0) - \sigma^2 t/2$ and variance $\sigma^2 t$. Then*

$$E[S(t) - K]_+ = S(0)\Phi(d_1) - K\Phi(d_2),$$

where

$$d_1 = \frac{1}{\sigma\sqrt{t}}\left(\ln S(0) - \ln K + \frac{\sigma^2 t}{2}\right),$$

$$d_2 = \frac{1}{\sigma\sqrt{t}}\left(\ln S(0) - \ln K - \frac{\sigma^2 t}{2}\right).$$

Proof. Substituting in Lemma 2 the mean $a = \ln S(0) - \frac{\sigma^2 t}{2}$, and variance $\sigma^2 t$, we get the result. □

2 Black Scholes formula for Asian option

European option depends on maturity time t. There exists a risk of price jumping at maturity. Asian option is more stable with respect to price jumping, because of the averaging feature.

For Asian options the payoff is determined by the average underlying price over some pre-set period of time at exercise.

One advantage of Asian options is that these reduce the risk of market manipulation of the underlying instrument at maturityan advantage for corporations.

The definition of call of Asian option is

$$P(T) = E[G(m,T) - K]_+$$

where

$$G(m,T) = \left[S(t_1) \cdot S(t_2) \cdots S_(t_m)\right]^{1/m}$$

and

$$T = t_1 + t_2 + ... + t_m.$$

In [1] can be found a Black Scholes formula for Asian option, but without proof. The next Theorem contains the detailed proof using Lemma 2.

Theorem 2 *Let $S(t)$ be a commodity price, such that $\ln S(t)$ has normal distribution with mean $\ln S(0) - \sigma^2 t/2$ and variance $\sigma^2 t$. Then the call for Asian option is*

$$E[G(m,T) - K]_+ = \exp\left(-\frac{m-1}{2m^2}T\sigma^2\right) S(0)\Phi(D_1) - K\Phi(d_1)$$

where:

$$D_1 = \frac{m}{\sigma\sqrt{T}}\left(\ln S(0) - \ln K + \frac{T\sigma^2}{2m^2}(2-m)\right)$$

$$d_1 = \frac{m}{\sigma\sqrt{T}}\left(\ln S(0) - \ln K - \frac{T\sigma^2}{2m}\right)$$

Proof. We have $\ln G(m,T) = [\ln S(t_1) + \ln S(t_2) + ... + \ln S(t_m)]/m$ and $\ln S(t) \sim N\left(\ln S(0) - \sigma^2 t/2, \sigma^2 t\right)$,then $\ln G(m,T) \sim N(M, V^2)$ where

$$M = \ln S(0) - \frac{\sigma^2}{2m}T, \tag{3}$$

$$V^2 = \frac{\sigma^2}{m^2}T. \tag{4}$$

Now we can apply lemma 2 for $\xi = \ln G(m,T)$;

$$E[G(m,T) - K]_+ = \exp\left(M + \frac{V^2}{2}\right)\Phi\left(\frac{M + V^2 - \ln K}{V}\right) - K\Phi\left(\frac{M - \ln K}{V}\right).$$

Substituting (3) and (4) in the above formula, we complete the proof of Theorem 2. □

3 Asian Options for k Several Futures

We have k months and the following N fixed time points.

First month: $t_1(1), t_1(2), ..., t_1(m(1))$;

Second month: $t_2(1), t_2(2), ..., t_2(m(2))$;

$$\cdots$$

k-th month: $t_k(1), t_k(2), ..., t_k(m(k))$;

Total

$$N = \sum_{i=1}^{k} m(i).$$

Let G be a geometric Asian option with N fixed time points:

$$G^N = \prod_{i=1}^{m(1)} S(t_1(i)) \cdot \prod_{i=1}^{m(2)} S(t_2(i)) \cdots \prod_{i=1}^{m(k)} S(t_k(i)). \tag{5}$$

We denote $r_j = \prod_{i=1}^{m(j)} S(t_j(i))$ then $\ln r_j = \sum_{i=1}^{m(j)} \ln S(t_j(i))$ and $\ln G = \frac{\sum_{j=1}^{k} \ln r_j}{N}$.

Then we can apply Lemma 2 for $\ln G$. Note that $NlnG$ has normal distribution with parameters:

$$N \ln G \sim \mathcal{N}(a_N, V^2)$$

Now we calculate the parameters a_N and V^2.

We have

$$\ln S(t_j(i)) \sim \mathcal{N}\left(\ln F_j - \frac{t_j(i)\sigma_j^2}{2},\ t_j(i)\sigma_j^2\right), \quad j = 1, 2, ..., k,$$

where $F_j = S(t_j(0))$.

Hence

$$\ln r_j \sim \mathcal{N}\left(m(j) \cdot \ln F_j - \sigma_j^2 T_j/2, \sigma_j^2 T_j\right)$$

where

$$T_j = \sum_{i=1}^{m(j)} t_j(i), \quad j = 1, 2, ..., k.$$

Thus

$$a_N = \sum_{j=1}^{k} \left(m(j) \ln F_j - \frac{T_j \sigma_j^2}{2}\right).$$

Also, when $\ln r_i$ and $\ln r_j$ are independent we can write:

$$V^2 = \sum_{j=1}^{k} \sigma_j^2 T_j.$$

But in the dependence case, we must continue in the next section.

4 Calculation of variance V^2

Let $S(t)$ be a stock price, i.e. a random process with *independent increments* such that $\ln S(t) - \ln S(0)$ has normal distribution with mean $-\sigma^2 t/2$ and standard deviation $\sigma\sqrt{t}$:

$$\ln S(t) \sim \mathcal{N}\left(\ln S(0) - \frac{t\sigma^2}{2},\ t\sigma^2\right).$$

We have the following recurrent formula

$$\ln S(t_i) = \ln S(t_{i-1}) - \frac{1}{2}\sigma^2(t_i - t_{i-1}) + w_i \sigma \sqrt{t_i - t_{i-1}}, \tag{6}$$

where the independent random variables w_i have normal distribution with zero mean and unit variance.

For k several futures the formula (6) has the form

$$\ln S_j(t_j(i)) = \ln S_j(t_j(i-1)) - \frac{1}{2}\sigma_j^2(t_j(i)-t_j(i-1)) + w_{ji}\sigma_j\sqrt{t_j(i)-t_j(i-1)},$$

$$i = 1, ..., m(j), \quad j = 1, 2, ..., k.$$

where the random variables w_{ji} have normal distribution with zero mean, unit variance and covariance:

$$E[w_{ji}w_{nl}] = \rho_{jn}, \quad j \neq n.$$

We consider the special case of uniform steps, where the time intervals $\Delta t = t_i - t_{i-1}$ are constant for each i.

As we know:

$$Var(\xi + \eta) = Var(\xi) + Var(\eta) + 2cov(\xi\eta).$$

So by recurrent formula we obtain:

$$Var\{\ln S_j(t_j(i)) - \ln S_j(t_j(i-1))\} = \sigma_j^2[t_j(i) - t_j(i-1)].$$

Thus we can calculate variance of sum to use the identity below:

$$\begin{aligned}&\ln S(t_1) + \ln S(t_2) + \ldots + \ln S(t_m) = \\ &\quad \ln(S(t_m)) - \ln(S(t_{m-1})) + 2[\ln(S(t_{m-1})) - \ln(S(t_{m-2}))] + \cdots \\ &\quad\quad m[\ln(S(t_1)) - \ln(S_0)] + m\ln(S_0),\end{aligned}$$

$$\begin{aligned}&\sum_i \ln S_j(t_j(i)) = [\ln S_j(t_j(m)) - \ln S_j(t_j(m-1))] + \\ &\quad 2[\ln S_j(t_j(m-1)) - \ln S_j(t_j(m-2))] + \cdots \\ &\quad\quad m[\ln S_j(t_j(1)) - \ln S_j(t_j(0))] + m\ln S_j(t_j(0))\end{aligned}$$

$$\begin{aligned}&N\ln G = \sum_{j=1}^{k}\sum_{i=1}^{m(j)} \ln S_j(t_j(i)) = \\ &\sum_{j=1}^{k}\sum_{i=1}^{m(j)}(m(j) - i + 1)[\ln S_j(t_j(i)) - \ln S_j(t_j(i-1))] + \sum_{j=1}^{k} m(j)\ln F_j\end{aligned}$$

where $F_j = E[S_j(t_j(0))]$.

For covariance we have:

$$cov[\ln S_j(t_j(i)) - \ln S_j(t_j(i-1)); \ln S_n(t_n(l)) - \ln S_n(t_n(l-1))] =$$

$$= \rho_{jn}\sigma_j\sqrt{t_j(i)-t_j(i-1)}\sigma_n\sqrt{t_n(l)-t_n(l-1)}. \qquad (7)$$

Now we can calculate desired variance:

$$V^2 = \sum_{j=1}^{k}\sum_{i=1}^{m(j)} Var\{(m(j)-i+1)[\ln S_j(t_j(i)) - \ln S_j(t_j(i-1))]\}+$$
$$2\sum_{j=1}^{k-1}\sum_{n=j+1}^{k}\sum_{i=1}^{m(j)}\sum_{l=1}^{m(n)} cov[\ln S_j(t_j(i)) - \ln S_j(t_j(i-1));$$
$$\ln S_n(t_n(l)) - \ln S_n(t_n(l-1))].$$

Using (7) we obtain:

$$V^2 = \sum_{j=1}^{k}\sum_{i=1}^{m(j)} \sigma_j^2 t_j(i)+$$
$$2\sum_{j=1}^{k-1}\sum_{n=j+1}^{k}\sum_{i=1}^{m(j)}\sum_{l=1}^{m(n)} \rho_{jn}(m(j)-i+1)\sigma_j\sqrt{t_j(i)-t_j(i-1)}\times$$
$$(m(n)-l+1)\sigma_n\sqrt{t_n(l)-t_n(l-1)}.$$

In the case of independence of w_{ji} and w_{nl} with $j \neq n$, we obtain $\rho_{jn} = 0$ and $V^2 = \sum_{j=1}^{k}\sigma_j^2 T_j$.

5 Numerical Experiments

We construct a computer program in Visual Basic to calculate incomes and damages using the obtained formulas. In the numerical examples we consider only single options, because the numerical estimate of unknown function ρ_{jn} is very complicated and unreliable.

The tables below are the short examples how the our program works. The detail tables are given in the paper [5]. In our example, we consider the European and Asian options for real oil prices and 7 selected initial times, K=65 , m=3 (*and first* t_0 *as 16th price of the market for computation of* μ *and* σ^2). We get the following numerical results.

Table 7.1

t_0	T	$S(0)$	$S(T)$	σ	$E[S(t)-K]_+$	$[S(T)-K]_+$
07/2/20	07/2/23	63.31	67.38	1.24	45.18	2.38
07/2/26	07/3/1	67.55	68.14	1.71	58.35	3.14
07/3/2	07/3/7	67.39	67.91	2.10	62.78	2.91
07/3/8	07/3/13	67.85	66.24	2.15	63.65	1.24
07/3/14	07/3/19	66.60	66.30	2.06	61.74	1.30
07/3/20	07/3/23	66.00	67.77	1.96	60.18	2.77
07/3/26	07/3/29	68.70	69.62	2.01	63.22	4.62

Table 7.2

t_0	T	$S(0)$	$S(T)$	σ	$G(m,T)$	$E[G(m,T)-K]_+$
07/2/20	07/2/23	63.31	67.38	1.24	66.13	2.77
07/2/26	07/3/1	67.55	68.14	1.71	67.98	1.16
07/3/2	07/3/7	67.39	67.91	2.10	67.02	0.37
07/3/8	07/3/13	67.85	66.24	2.15	66.50	0.32
07/3/14	07/3/19	66.60	66.30	2.06	66.14	0.40
07/3/20	07/3/23	66.00	67.77	1.96	67.39	0.54
07/3/26	07/3/29	68.70	69.62	2.01	69.40	0.51

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M. E. Alaei
Yerevan State University, Armenia
m_e_alaei@yahoo.com

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