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On Certain Subclass of Analytic Functions

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Abstract. In the present investigation, a new general subclass $M_{\alpha,\beta}(\phi)$ of analytic functions is defined. Some subordination relations, inclusion relations, integral preserving properties, convolution properties and some other interesting properties are studied.

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Introduction

Let A denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad z \in E. \tag{1}$$

which are analytic and univalent in the open unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$.

For two functions $f(z)$ and $g(z)$ analytic in E, we say that $f(z)$ is subordinate to $g(z)$, denoted by $f \prec g$ or $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ with $|w(z)| \leq |z|$ such that $f(z) = g(w(z))$. If $g(z)$ is univalent in E then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(E) \subset g(E)$. The idea of subordinations goes back to Lindelöf [16]. Subordination was more formally introduced and studied by Littelwood [19] and later by Rogosinski [17] and [18]. The concept of subordination was considered by Miller [11] and further investigated by Noor et al [15] and many others, see [12, 13, 16, 20]. Let $f, g \in A$ with $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, $z \in E$ and f is given by (1). Then the convolution (Hadamard product) of f and g is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \qquad z \in E. \tag{2}$$

We define the followings.

Definition 1 *Let P be the class of all functions ϕ which are analytic and univalent in E and satisfies $\phi(0) = 1$ and $\Re\phi(z) > 0$ which maps the open unit disc E onto a region starlike with respect to 1 and symmetric with respect to real axis.*

It can be easily verified that the class P is a convex set.

Definition 2 *Let $f \in A$, and $\frac{zf'(z)}{f(z)} \neq 0$ for $z \in E$. Let*

$$(1-\alpha)\left(\frac{f(z)}{z}\right)^{\beta} + \alpha f'(z)\left(\frac{f(z)}{z}\right)^{\beta-1} \prec \phi(z), \tag{3}$$

where $\alpha \geqq 0$, $\beta > 0$, $\phi \in P$ and the power are understood as principle value. Then f is said to belong to the class $M_{\alpha,\beta}(\phi)$.

It is obvious that the subclass $M_{1,\beta}\left(\frac{1+z}{1-z}\right)$ is the subclass of Bazilevic functions, which is the subclass of univalent functions S. The subclass $M_{1,\beta}\left(\frac{1+(1-2\rho)z}{1-z}\right)$ $(0 \leq \rho < 1)$ has been studied by Bazilevic [4], Singh [5], Owa [6], respectively. The subclass $M_{\alpha,1}\left(\frac{1+(1-2\rho)z}{1-z}\right)$ $(0 \leq \rho < 1)$ has been studied by Chichra [7], Ding, Ling and Bao [8], respectively. Chen and Liu [9] studied the subclass $M_{0,1}\left(\frac{1+(1-2\rho)z}{1-z}\right)$ $(0 \leq \rho < 1)$, and the subclass $M_{0,\beta}\left(\frac{1+(1-2\rho)z}{1-z}\right)$ $(0 \leq \rho < 1)$ has been studied by Liu [10], while the subclass $M_{\alpha,\beta}(p_k(z))$, for $k \geq 2$, $p_k(z)$ maps E onto the conic regions $\Omega_k = \left\{u + iv : u > k\sqrt{(u-1)^2 + v^2}\right\}$, have been studied by Noor [14]. The subclass $M_{\alpha,\beta}\left(\frac{1+Az}{1+Bz}\right)$ has been studied by Liu [2] and $M_{\alpha,\beta}(1+\lambda z) = M_{\alpha,\beta}(\lambda)$ has been studied by Ponnusamy [3].

In this article we shall study various properties of the class $M_{\alpha,\beta}(\phi)$. The results obtained generalize the related works of some authors. We obtained some new results too. To avoid repetition, it is admitted once that $\alpha \geq 0, \beta > 0, 0 < \delta \leq 1$ and $\theta > 0$. In order to derive our main results, we need to recall the following Lemmas.

1 Preliminaries Results

Lemma 1 *[1] Let the function $h(z)$ be convex univalent in E with*

$$h(0) = 1, \ \gamma \neq 0 \text{ and } Re\gamma > 0, \quad z \in E.$$

Suppose that the function

$$p(z) = 1 + p_1 z + p_2 z^2 + \ldots,$$

is analytic in E *and satisfy the following differential subordination*

$$p(z)+\frac{zp^{'}(z)}{\gamma}\prec h(z),\quad z\in E,$$

then

$$p(z)\prec q(z)\prec h(z),\quad z\in E,$$

where

$$q\left(z\right)=\frac{\gamma}{z^{\gamma}}\int_{0}^{z}h\left(t\right)t^{\gamma-1}dt.$$

The function $q(z)$ *is convex and is the best dominant.*

Lemma 2 *[11] Let* $0<\lambda_1<\lambda<1$ *and let* $Q\in H$ *satisfy*

$$Q\left(z\right)\prec 1+\lambda_1 z,\qquad Q\left(0\right)=1.$$

(i) If $p\in H$, $p\left(0\right)=1$, *and*

$$Q\left(z\right)\left[\gamma+\left(1-\gamma\right)p\left(z\right)\right]\prec 1+\lambda z,$$

where

$$\gamma=\begin{cases}\frac{1-\lambda}{1+\lambda_1}, & 0<\lambda+\lambda_1\leq 1;\\ \\ \frac{1-\left(\lambda^2+\lambda_1^2\right)}{2\left(1-\lambda_1^2\right)}, & \lambda_1^2+\lambda^2\leq 1\leq\lambda+\lambda_1,\end{cases}\tag{4}$$

then $\Re p\left(z\right)>0$.
(ii) If $w\in H$, $w\left(0\right)=0$, *and*

$$Q\left(z\right)\left[1+w\left(z\right)\right]\prec 1+\lambda z,$$

then

$$\left|w\left(z\right)\leq\frac{\lambda+\lambda_1}{1-\lambda_1}\right|=\rho\leq 1\qquad \text{if}\quad \lambda+2\lambda_1\leq 1.\tag{5}$$

The bound (5) *and the value of* γ *given by* (4) *are the best possible.*

Lemma 3 *[22] Let* $p\left(z\right)$ *be an analytic function in* E *with* $p\left(0\right)=1$ *and* $p\left(z\right)\neq 0$ $\left(z\in E\right)$. *If there exists a point* $z_\circ\in E$, *such that,*

$$\left|\arg p(z)\right|<\frac{\pi}{2}\theta\quad \text{for}\quad \left|z\right|<\left|z_\circ\right|,$$

and

$$\left|\arg p(z_\circ)\right|=\frac{\pi}{2}\theta,$$

for some $\theta>0$, *then we have*

$$\frac{z_\circ p'(z_\circ)}{p(z_\circ)}=ik\theta,$$

where $k\geq 1$, *when* $\arg p(z_\circ)=\frac{\pi}{2}\theta$ *and* $k\leq -1$ *when* $\arg p(z_\circ)=-\frac{\pi}{2}\theta$.

Lemma 4 *[23]If $p(z)$ is analytic in E, $p(0)=1$ and $Re(p(z)>\frac{1}{2}$, $z\in E$ then for any function F analytic in E, the function $p*F$ takes values in the convex hull of the image of E under F.*

2 Main Results

Theorem 1 *Let $f\in M_{\alpha,\beta}(\phi)$. Then $f\in M_{0,\beta}(\phi)$ in E.*

Proof. Let $p(z)=\left(\frac{f(z)}{z}\right)^{\beta}$. We note that p is analytic with $p(0)=1$. The case $\alpha=0$ is trivial. So we suppose $\alpha>0$. Now

$$(1-\alpha)\left(\frac{f(z)}{z}\right)^{\beta}+\alpha f'(z)\left(\frac{f(z)}{z}\right)^{\beta-1}=p(z)+\frac{\alpha}{\beta}zp'(z).$$

Since $f\in M_{\alpha,\beta}(\phi)$, it follows from the definition that

$$p(z)+\frac{\alpha}{\beta}zp'(z)\prec\phi(z),$$

and applying Lemma 1, we have

$$p(z)\prec q(z)\prec\phi(z),\qquad z\in E,$$

where q is the best dominant given by

$$q(z)=\frac{\beta}{\alpha}z^{-\frac{\beta}{\alpha}}\int_0^z t^{\frac{\beta}{\alpha}-1}\phi(t)\,dt. \tag{6}$$

This proves that $f\in M_{0,\beta}(\phi)$ in E. □

We have the following special case obtained by Liu [2].

Corollary 1 *For $\beta>0,\alpha\geq 0$, $\phi(z)=\dfrac{1+Az}{1+Bz}$, $-1\leq B<A\leq 1$, the following inclusion holds*

$$M_{\alpha,\beta}[A,B]\subset M_{0,\beta}[A,B].$$

Theorem 2 *For each $\beta>0$, $0\leq\alpha_1\leq\alpha_2$, we have*

$$M_{\alpha_2,\beta}(\phi)\subset M_{\alpha_1,\beta}(\phi)\qquad \text{in } E.$$

Proof. Let $f\in M_{\alpha_2,\beta}(\phi)$. Then

$$J(\alpha_2,\beta,f)=(1-\alpha_2)\left(\frac{f(z)}{z}\right)^{\beta}+\alpha_2 f'(z)\left(\frac{f(z)}{z}\right)^{\beta-1}\prec\phi(z).$$

Now

$$(1-\alpha_1)\left(\frac{f(z)}{z}\right)^{\beta}+\alpha_1 f'(z)\left(\frac{f(z)}{z}\right)^{\beta-1} = \left(1-\frac{\alpha_1}{\alpha_2}\right)J(\alpha_2,\beta,f)+\frac{\alpha_1}{\alpha_2}\left(\frac{f(z)}{z}\right)^{\beta} = \left(1-\frac{\alpha_1}{\alpha_2}\right)p_1(z)+p_2(z)\frac{\alpha_1}{\alpha_2}.$$

Since $p_1, p_2 \in P$ and P is a convex set, we obtain the required result that $f \in M_{\alpha_1,\beta}(\phi)$ in E. □

Theorem 3 *Let* $f \in M_{\alpha,\beta}(\phi)$ *for* $\alpha > 0$, $\beta > 0$. *Let*

$$F(z)=\left[\frac{1}{\alpha}z^{\left(\beta-\frac{1}{\alpha}\right)}\int_0^z t^{\frac{1}{\alpha}-1-\beta}f^{\beta}(t)\,dt\right]^{\frac{1}{\beta}}.$$

Then $F \in M_{\alpha,\beta}(\phi)$ *in* E.

Proof. We have

$$\left[\alpha z^{\left(\frac{1}{\alpha}-\beta\right)}F^{\beta}(z)\right]'=z^{\frac{1}{\alpha}-1-\beta}f^{\beta}(z).$$

This implies

$$(1-\alpha\beta)\left(\frac{F(z)}{z}\right)^{\beta}+\alpha\beta F'(z)\left(\frac{F(z)}{z}\right)^{\beta-1}=\left(\frac{f(z)}{z}\right)^{\beta}.$$

Now the result follows from Theorem 1. □

We have the following special case obtained by Noor [14].

Corollary 2 *Let* $f \in M_{\alpha,\beta}(P_k(z))$ *for* $\alpha > 0$, $\beta > 0$. *Let*

$$F(z)=\left[\frac{1}{\alpha}z^{\left(\beta-\frac{1}{\alpha}\right)}\int_0^z t^{\frac{1}{\alpha}-1-\beta}f^{\beta}(t)\,dt\right]^{\frac{1}{\beta}}.$$

Then $F \in M_{\alpha,\beta}(P_k(z))$ *in* E.

Theorem 4 *Let* $f \in M_{\alpha,\beta}(1+\lambda z)$. *Then* $f \in S^*(1-\alpha)$.

Proof. Let

$$p(z) = \left(\frac{f(z)}{z}\right)^{\beta}.$$

Then

$$(1-\alpha)\left(\frac{f(z)}{z}\right)^{\beta} + \alpha f'(z)\left(\frac{f(z)}{z}\right)^{\beta-1} = p(z) + \frac{zp'(z)}{\frac{\beta}{\alpha}},$$

and

$$p(z) + \frac{zp'(z)}{\gamma} \prec 1 + \lambda z, \qquad \gamma = \frac{\beta}{\alpha}.$$

Now

$$p(z) + \frac{zp'(z)}{\gamma} = p(z) * \phi(z),$$

where

$$\phi(z) = 1 + \sum_{n=1}^{\infty}\left(1 + \frac{n}{\gamma}\right)z^{n}.$$

Since

$$\phi^{-1}(z) = 1 + \sum_{n=1}^{\infty}\frac{z^{n}}{\left(1 + \frac{n}{\gamma}\right)}$$

is convex for $\gamma > 0$ and

$$\phi(z) * \phi^{-1}(z) = \sum_{n=1}^{\infty} z^{n},$$

it implies

$$\begin{aligned} p(z) &\prec \gamma z^{-\gamma}\int_{0}^{z}(1+\lambda z)\, t^{\gamma-1}dt \\ &= 1 + \frac{\lambda\gamma}{\gamma+1}z, \end{aligned}$$

that is

$$p(z) \prec 1 + \lambda_1 z.$$

Further, we can write

$$\left(\frac{f(z)}{z}\right)^{\beta}\left\{(1-\alpha) + \alpha\frac{zf'(z)}{f(z)}\right\} \prec 1 + \lambda z.$$

That is

$$p(z)\left\{(1-\alpha) + \alpha p_1(z)\right\} \prec 1 + \lambda z,$$

where

$$p(z) \prec 1+\lambda_1 z, \qquad p_1(z) = \frac{zf'(z)}{f(z)}.$$

We now use the Lemma 2, to have

$$\Re p_1(z) > 0,$$

and consequently $f \in S^*(1-\alpha)$, where $(1-\alpha) = \gamma$ is given in (4). $\square$

Theorem 5 *Let $f \in M_{\alpha,\beta}(\phi)$ in E. Let for $c > 0$, F be defined as*

$$F(z) = \left[\frac{\beta+c}{z^c}\int_0^z t^{c-1} f^{\beta}(t)\,dt\right]^{\frac{1}{\beta}}. \tag{7}$$

Then $F \in M_{\alpha_1,\beta}(\phi_1)$, where $\alpha_1 = \frac{\beta}{\beta+c}$, and ϕ_1 is given as

$$\phi_1(z) = \frac{\beta}{\alpha} z^{-\frac{\beta}{\alpha}} \int_0^z t^{\frac{\beta}{\alpha}-1}\phi(t)\,dt.$$

Proof. From (7), we have with some computations

$$\frac{c}{\beta+c}\left(\frac{F(z)}{z}\right)^{\beta} + \frac{\beta}{\beta+c}F'(z)\left(\frac{F(z)}{z}\right)^{\beta-1} = \left(\frac{f(z)}{z}\right)^{\beta}.$$

Since $f \in M_{\alpha,\beta}(\phi)$, it follows that

$$\left(\frac{f(z)}{z}\right)^{\beta} \prec \phi_1(z) \prec \phi(z),$$

with

$$\phi_1(z) = \frac{\beta}{\alpha} z^{-\frac{\beta}{\alpha}} \int_0^z t^{\frac{\beta}{\alpha}-1}\phi(t)\,dt.$$

Hence for $\alpha_1 = \frac{\beta}{\beta+c}$, we have

$$\frac{c}{\beta+c}\left(\frac{F(z)}{z}\right)^{\beta} + \frac{\beta}{\beta+c}F'(z)\left(\frac{F(z)}{z}\right)^{\beta-1} \prec \phi_1(z),$$

which proves $F \in M_{\alpha_1,\beta}(\phi_1)$ in E. $\square$

Theorem 6 *Let $f \in M_{\alpha,1}(\phi)$ and $\Re\left(\frac{\varphi(z)}{z}\right) > \frac{1}{2}$ for $z \in E$. Then*

$$h(z) = (\varphi * f)(z) \in M_{\alpha,1}(\phi).$$

Proof. Since $f \in M_{\alpha,1}(\phi)$, we can write

$$\begin{aligned} & (1-\alpha)\frac{h(z)}{z}+\alpha h'(z) \\ = \; & \frac{\varphi(z)}{z} * \left((1-\alpha)\frac{f(z)}{z}+\alpha f'(z)\right). \end{aligned} \tag{8}$$

From hypothesis $f \in M_{\alpha,1}(\phi)$ and $\Re\left(\frac{\varphi(z)}{z}\right) > \frac{1}{2}$ in $z \in E$, therefore it follows from (8) by using Lemma 4 that $h(z) = (\varphi * f)(z) \in M_{\alpha,1}(\phi)$. This completes the proof of Theorem 6. □

Theorem 7 *Let $f \in A$, with $\frac{zf'(z)}{f(z)} \neq 0$. Suppose $\phi(z) = \left(\frac{1+z}{1-z}\right)^{\delta}$, if*

$$(1-\alpha)\left(\frac{f(z)}{z}\right)^{\beta} + \alpha f'(z)\left(\frac{f(z)}{z}\right)^{\beta-1} \prec \left(\frac{1+z}{1-z}\right)^{\delta}, \tag{9}$$

where

$$\delta = \delta(\alpha,\beta,\theta) = \frac{\pi}{2}\left(\theta + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha\theta}{\beta}\right)\right), \tag{10}$$

then

$$\left(\frac{f(z)}{z}\right)^{\beta} \prec \left(\frac{1+z}{1-z}\right)^{\theta}. \tag{11}$$

Proof. Let

$$\left(\frac{f(z)}{z}\right)^{\beta} = p(z), \tag{12}$$

where $p(z)$ is analytic in E and $p(0) = 1$. Differentiating (12) and simplifying we have

$$(1-\alpha)\left(\frac{f(z)}{z}\right)^{\beta} + \alpha f'(z)\left(\frac{f(z)}{z}\right)^{\beta-1} = p(z) + \frac{\alpha}{\beta}zp'(z).$$

Let

$$h(z) = \left(\frac{1+z}{1-z}\right)^{\delta} \quad \text{and} \quad q(z) = \left(\frac{1+z}{1-z}\right)^{\theta},$$

then

$$|\arg h(z)| < \frac{\delta\pi}{2} \quad \text{and} \quad |\arg q(z)| < \frac{\theta\pi}{2}.$$

Suppose that $p(z) \nprec q(z)$. Since $p(0) = q(0) = 1$ and $p(z) \neq 0$, there exists a point $z_\circ \in E$ such that

$$|\arg p(z)| < \frac{\theta\pi}{2}\;(|z| < |z_\circ|) \quad \text{and} \quad |\arg p(z_\circ)| = \frac{\theta\pi}{2}.$$

(i) for the case $\arg p(z_\circ) = \frac{\pi}{2}\theta$, we have

$$\begin{aligned}\arg\left(p(z_\circ) + \frac{\alpha}{\beta} z_\circ p'(z_\circ)\right) &= \arg p(z_\circ) + \arg\left(1 + \frac{\alpha}{\beta}\frac{z_\circ p'(z_\circ)}{p(z_\circ)}\right) \\ &= \arg(i\theta)^\theta + \arg\left(\frac{\beta + i\alpha k\theta}{\beta}\right) \\ &\geq \frac{\pi}{2}\theta + \tan^{-1}\left(\frac{\alpha\theta}{\beta}\right).\end{aligned}$$

This contradicts our condition in the theorem.

(ii) for the case $\arg p(z_\circ) = -\frac{\pi}{2}\theta$, the application of the same method as in (i) shows that

$$\arg\left(p(z_\circ) + \frac{\alpha}{\beta} z_\circ p'(z_\circ)\right) \leq -\left(\frac{\pi}{2}\theta + \tan^{-1}\left(\frac{\alpha\theta}{\beta}\right)\right).$$

This also contradicts the assumption of the theorem, hence the theorem is proved. □

Putting $\alpha = 1$, we have the following special case obtained by Lashin [21].

Corollary 3 *Let* $f \in A$, *and* $\frac{zf'(z)}{f(z)} \neq 0$. *Suppose* $\phi(z) = \left(\frac{1+z}{1-z}\right)^\delta$, *if*

$$f'(z)\left(\frac{f(z)}{z}\right)^{\beta-1} \prec \left(\frac{1+z}{1-z}\right)^\delta,$$

where

$$\delta = \delta(\beta, \theta) = \frac{\pi}{2}\left(\theta + \frac{2}{\pi}\tan^{-1}\left(\frac{\theta}{\beta}\right)\right),$$

then

$$\left(\frac{f(z)}{z}\right)^\beta \prec \left(\frac{1+z}{1-z}\right)^\theta.$$

Putting $\beta = 1$ *we have the following special case obtained by Lashin [21].*

Corollary 4 *Let* $f \in A$, *and* $\frac{zf'(z)}{f(z)} \neq 0$. *Suppose* $\phi(z) = \left(\frac{1+z}{1-z}\right)^\delta$, *if*

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) \prec \left(\frac{1+z}{1-z}\right)^\delta,$$

where

$$\delta = \delta(\beta, \theta) = \frac{\pi}{2}\left(\theta + \frac{2}{\pi}\tan^{-1}(\alpha\theta)\right),$$

then

$$\frac{f(z)}{z} \prec \left(\frac{1+z}{1-z}\right)^\theta.$$

Theorem 8 *Let $f \in A$, and $\frac{zf'(z)}{f(z)} \neq 0$. Suppose $\phi(z) = \left(\frac{1+z}{1-z}\right)^{\delta}$, if*

$$\left(\frac{f(z)}{z}\right)^{\beta}\left((1-\alpha)\left(\frac{f(z)}{z}\right)^{\beta} + \alpha f'(z)\left(\frac{f(z)}{z}\right)^{\beta-1}\right) \prec \left(\frac{1+z}{1-z}\right)^{\delta},$$

where

$$\delta = \delta(\alpha, \beta, \theta) = \frac{\pi}{2}\left(2\theta + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha\theta}{\beta}\right)\right),$$

then

$$\left(\frac{f(z)}{z}\right)^{\beta} \prec \left(\frac{1+z}{1-z}\right)^{\theta}.$$

Proof. Let

$$\left(\frac{f(z)}{z}\right)^{\beta} = p(z), \tag{13}$$

where $p(z)$ is analytic in E and $p(0) = 1$. Differentiation of (13) and simple computation gives

$$\left(\frac{f(z)}{z}\right)^{\beta}\left((1-\alpha)\left(\frac{f(z)}{z}\right)^{\beta} + \alpha f'(z)\left(\frac{f(z)}{z}\right)^{\beta-1}\right) = p(z)\left(p(z) + \frac{\alpha}{\beta}zp'(z)\right).$$

Let

$$h(z) = \left(\frac{1+z}{1-z}\right)^{\delta} \quad \text{and} \quad q(z) = \left(\frac{1+z}{1-z}\right)^{\theta},$$

so that

$$|\arg h(z)| < \frac{\delta\pi}{2} \quad \text{and} \quad |\arg q(z)| < \frac{\theta\pi}{2}.$$

Suppose that $p(z) \not\prec q(z)$, noting that $p(0) = q(0) = 1$ and $p(z) \neq 0$, there exists a point $z_\circ \in E$ such that

$$|\arg p(z)| < \frac{\theta\pi}{2}\ (|z| < |z_\circ|) \quad \text{and} \quad |\arg p(z_\circ)| = \frac{\theta\pi}{2}.$$

(i) for the case $\arg p(z_\circ) = \frac{\pi}{2}\theta$, we have

$$\begin{aligned}
\arg\left(p(z_\circ)\left(p(z_\circ) + \frac{\alpha}{\beta}z_\circ p'(z_\circ)\right)\right) &= 2\arg p(z_\circ) + \arg\left(1 + \frac{\alpha}{\beta}\frac{z_\circ p'(z_\circ)}{p(z_\circ)}\right) \\
&= 2\arg(i\theta)^{\theta} + \arg\left(\frac{\beta + i\alpha k\theta}{\beta}\right) \\
&\geq \frac{\pi}{2}\left(2\theta + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha\theta}{\beta}\right)\right).
\end{aligned}$$

This contradicts our condition in the theorem.

(ii) for the case $\arg p(z_\circ) = -\frac{\pi}{2}\theta$, the application of the same method as in (i) shows that

$$\arg\left(p(z_\circ)\left(p(z_\circ) + \frac{\alpha}{\beta} z_\circ p'(z_\circ)\right)\right) \leq -\frac{\pi}{2}\left(2\theta + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha\theta}{\beta}\right)\right).$$

This also contradicts the assumption of the theorem, hence the theorem is proved. □

Putting $\alpha = 1$, we have the following special case.

Corollary 5 *Let $f \in A$, and $\frac{zf'(z)}{f(z)} \neq 0$. Suppose $\phi(z) = \left(\frac{1+z}{1-z}\right)^\delta$, if*

$$f'(z)\left(\frac{f(z)}{z}\right)^{2\beta-1} \prec \left(\frac{1+z}{1-z}\right)^\delta,$$

where

$$\delta = \delta(\beta, \theta) = \frac{\pi}{2}\left(2\theta + \frac{2}{\pi}\tan^{-1}\left(\frac{\theta}{\beta}\right)\right),$$

then

$$\left(\frac{f(z)}{z}\right)^\beta \prec \left(\frac{1+z}{1-z}\right)^\theta.$$

Theorem 9 *Let $f \in A$, and $\frac{zf'(z)}{f(z)} \neq 0$. Suppose $\phi(z) = \left(\frac{1+z}{1-z}\right)^\delta$, if*

$$(1-\alpha)\left(\frac{f(z)}{z}\right)^\beta + \alpha f'(z)\left(\frac{f(z)}{z}\right)^{\beta-1} \prec \left(\frac{1+z}{1-z}\right)^\delta,$$

where

$$\delta = \delta(\alpha, \beta, \theta) = \frac{\pi}{2}\left(\beta\theta + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha\theta}{\beta}\right)\right),$$

then

$$\frac{f(z)}{z} \prec \left(\frac{1+z}{1-z}\right)^\theta.$$

Proof. Let

$$\frac{f(z)}{z} = p(z), \tag{14}$$

where $p(z)$ is analytic in E and $p(0) = 1$. The remaining part of the proof of Theorem 9 is similar to that of Theorem 7 and so we omit it. □

Putting $\beta = 1$ we have the following special case obtained by Lashin [21].

Corollary 6 *Let* $f \in A$, *and* $\frac{zf'(z)}{f(z)} \neq 0$. *Suppose* $\phi(z) = \left(\frac{1+z}{1-z}\right)^{\delta}$, *if*

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) \prec \left(\frac{1+z}{1-z}\right)^{\delta},$$

where

$$\delta = \delta(\alpha, \theta) = \frac{\pi}{2}\left(\theta + \frac{2}{\pi}\tan^{-1}(\alpha\theta)\right),$$

then

$$\frac{f(z)}{z} \prec \left(\frac{1+z}{1-z}\right)^{\theta}.$$

Theorem 10 *Let* $f \in A$, *and* $\frac{zf'(z)}{f(z)} \neq 0$. *Suppose* $\phi(z) = \left(\frac{1+z}{1-z}\right)^{\delta}$, *if*

$$\left(\frac{f(z)}{z}\right)^{\beta}\left((1-\alpha)\left(\frac{f(z)}{z}\right)^{\beta} + \alpha f'(z)\left(\frac{f(z)}{z}\right)^{\beta-1}\right) \prec \left(\frac{1+z}{1-z}\right)^{\delta},$$

where

$$\delta = \delta(\alpha, \beta, \theta) = \frac{\pi}{2}\left(2\beta\theta + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha\theta}{\beta}\right)\right),$$

then

$$\frac{f(z)}{z} \prec \left(\frac{1+z}{1-z}\right)^{\theta}.$$

Proof. Let

$$\frac{f(z)}{z} = p(z),$$

where $p(z)$ is analytic in E and $p(0) = 1$. The remaining part of the proof of Theorem 10 is similar to that of Theorem 8 and so we omit it. □

Putting $\alpha = 1$, we have the following special case.

Corollary 7 *Let* $f \in A$, *and* $\frac{zf'(z)}{f(z)} \neq 0$. *Suppose* $\phi(z) = \left(\frac{1+z}{1-z}\right)^{\delta}$, *if*

$$f'(z)\left(\frac{f(z)}{z}\right)^{2\beta-1} \prec \left(\frac{1+z}{1-z}\right)^{\delta},$$

where

$$\delta = \delta(\beta, \theta) = \frac{\pi}{2}\left(2\beta\theta + \frac{2}{\pi}\tan^{-1}\left(\frac{\theta}{\beta}\right)\right),$$

then

$$\left(\frac{f(z)}{z}\right)^{\beta} \prec \left(\frac{1+z}{1-z}\right)^{\theta}.$$

Conclusion

In this paper, we have used the techniques of differential subordination and convolution to obtain inclusion theorems and subordination theorems. Many interesting particular cases of the main theorems are emphasized in the form of corollaries. The ideas and techniques of this work can motivate and inspire others to further explore this interesting field.

References

[1] P. Eenigenburg, S. S. Miller, P. T. Mocanu and M. O. Reade, *On a Briot-Bouquet differential subordination*, Rev. Roum. Math. Pures Appl., **29**(1984), 567-573.

[2] M. Liu, *On certain subclasses of p-valent functions*, Soochow J. Math., **26**(2000), 163-171.

[3] S. Ponnusamy and S. Rajasekaran, *New sufficient conditions for starlike and univalent functions*, Soochow J. Math., **21**(1995), 193-201.

[4] E. Bazilevic, *On a case of integrability in quadratures of the Loewner-Kuarev equation*, Mat. Sb., **37**(1955), 471-476.

[5] R. Singh, *On Bazilevic functions*, Proc. Amer. Math. Sci., **38**(1973), 261-267.

[6] S. Owa and M. Obradovic, *Certain subclasses of Bazilevic functions of type α* , Int. J. Math. Math. Sci., **9**(1986), 97-105.

[7] P. N. Chichra, *New subclass of the class of close-to-convex functions*, Proc. Amer. Math. Soc., **62**(1977), 37-43.

[8] S. S. Ding, Y. Ling and G. J. Bao, *Some properties of a class of analytic functions*, J. Math. Anal. Appl., **195**(1995), 71-81.

[9] M. P. Chen, *On the regular functions satisfying $\Re[\frac{f(z)}{z}] > \rho$* , Bull. Inst. Math. Acad. Sinica., **3**(1975), 65-70.

[10] M. S. Liu, *The radius of univalence for certain class of analytic functions*, In: Lu Jianke and Wen Guochun eds, boundary value problems, integral equations and related problems, Singapore: World Sci. Publishing., (2000), 122-128.

[11] S. S. Miller and P. T. Mocanu, *Differential subordination and univalent functions,* Michigan Math. J., **28**(1981), 157-172.

[12] N. Khan, B. Khan, Q. Z. Ahmad and S. Ahmad, *Some convolution properties of multivalent analytic functions,* AIMS Mathematics, **2(**2017**)**, 260-268.

[13] K. I. Noor, N. Khan and Q. Z. Ahmad, *Some properties of multivalent spiral-like functions,* Maejo Int. J. Sci. Technol., **12(**2018), 139-151.

[14] K. I. Noor, *On certain classes of analytic functions*, J. Ineq. Pure. Appl. Math., **7**(2006), 1-5.

[15] K. I. Noor, S. N. Malik, M. Arif and M. Raza, *On bounded boundary and bounded radius rotation related with Janowski function,* World Appl. Sci. J., **12**(2011), 895-902.

[16] E. Lindelöf, *Mémorie sur certaines inequalitiés dans la théorie des fonctions monogènes et sur quelques propiétés nouvelles de ces fonctions dans le voisinage d'un point singulier essentiel*, Acta Soc. Sci. Fenn., **35**(1908), 1-35.

[17] W. Rogosinski, *On subordinate functions*, Proc. Camb. Phil. Soc., **35**(1939), 1-26.

[18] W. Rogosinski, *On the coefficients of subordinate functions*, Proc. London Math. Soc., **48**(1943), 48-82.

[19] J. E. Littelwood, *On inequalities in the theory of functions*, Proc. London Math. Soc., **23**(1925), 481-519.

[20] M. A. Rosihan and V. Ravichandran, *Convolution and differential subordination for multivalent functions,* Bull. Malays. Math. Sci., **32**(2009), 351-360.

[21] A. Y. Lashin, *Application of Nunokawa's theorem,* J. Pure and Appl. Math., **5**(2004), 1-11.

[22] M. Nunokawa, *On the order of strongly convex functions,* Proc. Japan Acad. Ser. A. Math. Soc., **69**(1993), 234-237.

[23] R. Singh and S. Singh, *Convolution properties of a class of starlike functions*, Proc. Amer. Math. Soc., **108**(1989), 145-152.

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