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Lebesgue Integral Inequalities of Jensen Type for λ-Convex Functions

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Abstract. Some Lebesgue integral inequalities of Jensen type for λ-convex functions defined on real intervals are given.

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Introduction

0.1 h-Convex Functions

We recall here some concepts of convexity that are well known in the literature.

Let I be an interval in $\mathbb{R}$.

Definition 1 ([42]) *We say that $f : I \to \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0,1)$ we have*

$$f(tx + (1-t)y) \le \frac{1}{t} f(x) + \frac{1}{1-t} f(y). \tag{1}$$

Some further properties of this class of functions can be found in [32], [33], [35], [48], [51] and [52]. Among others, its has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

The above concept can be extended for functions $f : C \subseteq X \to [0,\infty)$ where C is a convex subset of the real or complex linear space X and the inequality (1) is satisfied for any vectors $x, y \in C$ and $t \in (0,1)$. If the function $f : C \subseteq X \to \mathbb{R}$ is non-negative and convex, then it is of Godunova-Levin type.

Definition 2 (**[35]**) *We say that a function $f : I \to \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in [0,1]$ we have*

$$f(tx + (1-t)y) \leq f(x) + f(y). \tag{2}$$

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contains all nonnegative monotone, convex and *quasi convex functions,* i. e. nonnegative functions satisfying

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\} \tag{3}$$

for all $x, y \in I$ and $t \in [0,1]$.

For some results on P-functions see [35] and [49], while for quasi convex functions the reader can consult [34].

If $f : C \subseteq X \to [0, \infty)$, where C is a convex subset of the real or complex linear space X, then we say that it is of P-type (or quasi-convex) if the inequality (2) (or (3)) holds true for $x, y \in C$ and $t \in [0,1]$.

Definition 3 (**[7]**) *Let s be a real number, $s \in (0,1]$. A function $f : [0,\infty) \to [0,\infty)$ is said to be s-convex (in the second sense) or Breckner s-convex if*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0,\infty)$ and $t \in [0,1]$.

For some properties of this class of functions see [1], [2], [7], [8], [30], [31], [43], [45] and [54].

The concept of Breckner s-convexity can be similarly extended for functions defined on convex subsets of linear spaces.

It is well known that if $(X, \|\cdot\|)$ is a normed linear space, then the function $f(x) = \|x\|^p, p \geq 1$ is convex on X.

Utilising the elementary inequality $(a+b)^s \leq a^s + b^s$ that holds for any $a, b \geq 0$ and $s \in (0,1]$, we have for the function $g(x) = \|x\|^s$

$$\begin{aligned} g(tx + (1-t)y) &= \|tx + (1-t)y\|^s \leq (t\|x\| + (1-t)\|y\|)^s \\ &\leq (t\|x\|)^s + [(1-t)\|y\|]^s \\ &= t^s g(x) + (1-t)^s g(y) \end{aligned}$$

for any $x, y \in X$ and $t \in [0,1]$, which shows that g is Breckner s-convex on X.

In order to unify the above concepts for functions of real variable, S. Varošanec introduced the concept of h-convex function as follows.

Assume that I and J are intervals in $\mathbb{R}, (0,1) \subseteq J$ and functions h and f are real non-negative functions defined in J and I, respectively.

Definition 4 ([58]) *Let $h : J \to [0, \infty)$ with h not identical to 0. We say that $f : I \to [0, \infty)$ is an h-convex function if for all $x, y \in I$ we have*

$$f(tx + (1-t)y) \leq h(t) f(x) + h(1-t) f(y) \tag{4}$$

for all $t \in (0,1)$.

For some results concerning this class of functions see [58], [6], [46], [55], [53] and [57].

This concept can be extended for functions defined on convex subsets of linear spaces in the same way as above replacing the interval I be the corresponding convex subset C of the linear space X.

Now we can introduce another class of functions.

Definition 5 *We say that the function $f : C \subseteq X \to [0, \infty)$ is of s-Godunova-Levin type, with $s \in [0,1]$, if*

$$f(tx + (1-t)y) \leq \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y), \tag{5}$$

for all $t \in (0,1)$ and $x, y \in C$.

We observe that for $s = 0$ we obtain the class of P-functions while for $s = 1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(C)$ the class of s-Godunova-Levin functions defined on C, then we obviously have

$$P(C) = Q_0(C) \subseteq Q_{s_1}(C) \subseteq Q_{s_2}(C) \subseteq Q_1(C) = Q(C)$$

for $0 \leq s_1 \leq s_2 \leq 1$.

For different inequalities related to these classes of functions, see [1]-[4], [6], [9]-[41], [44]-[46] and [49]-[57].

A function $h : J \to \mathbb{R}$ is said to be *supermultiplicative* if

$$h(ts) \geq h(t) h(s) \text{ for any } t, s \in J. \tag{6}$$

If the inequality (6) is reversed, then h is said to be *submultiplicative*. If the equality holds in (6) then h is said to be a multiplicative function on J.

In [58] it has been noted that if $h : [0, \infty) \to [0, \infty)$ with $h(t) = (x+c)^{p-1}$, then for $c = 0$ the function h is multiplicative. If $c \geq 1$, then for $p \in (0,1)$ the function h is supermultiplicative and for $p > 1$ the function is submultiplicative.

We observe that, if h, g are nonnegative and supermultiplicative, then their product is alike. In particular, if h is supermultiplicative then its product with a power function $\ell_r(t) = t^r$ is also supermultiplicative.

We recall the following Hermite-Hadamard type inequality for h-convex functions from [53]:

Theorem 1 *Let $f : I \to [0,\infty)$ be an integrable h-convex function on I and a, $b \in I$ with $a < b$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le [f(a)+f(b)]\int_0^1 h(t)\,dt, \qquad (7)$$

provided $\int_0^1 h(t)\,dt < \infty$.

0.2 λ-Convex Functions

We start with the following definition (see also [26]):

Definition 6 *Let $\lambda : [0,\infty) \to [0,\infty)$ be a function with the property that $\lambda(t) > 0$ for all $t > 0$. A mapping $f : C \to \mathbb{R}$ defined on convex subset C of a linear space X is called λ-convex on C if*

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right) \le \frac{\lambda(\alpha) f(x) + \lambda(\beta) f(y)}{\lambda(\alpha+\beta)} \qquad (8)$$

for all $\alpha, \beta \ge 0$ with $\alpha + \beta > 0$ and $x, y \in C$.

We observe that if $f : C \to \mathbb{R}$ is λ-convex on C, then f is h-convex on C with $h(t) = \frac{\lambda(t)}{\lambda(1)}$, $t \in [0,1]$.

If $f : C \to [0,\infty)$ is h-convex function with h supermultiplicative on $[0,\infty)$, then f is λ-convex with $\lambda = h$.

Indeed, if $\alpha, \beta \ge 0$ with $\alpha + \beta > 0$ and $x, y \in C$ then

$$\begin{aligned} f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right) &\le h\left(\frac{\alpha}{\alpha+\beta}\right) f(x) + h\left(\frac{\beta}{\alpha+\beta}\right) f(y) \\ &\le \frac{h(\alpha) f(x) + h(\beta) f(y)}{h(\alpha+\beta)}. \end{aligned}$$

The following proposition contains some properties of λ-convex functions [26].

Proposition 1 *Let $f : C \to \mathbb{R}$ be a λ-convex function on C.*

(i) If $\lambda(0) > 0$, then we have $f(x) \ge 0$ for all $x \in C$;

(ii) If there exists $x_0 \in C$ so that $f(x_0) > 0$, then

$$\lambda(\alpha+\beta) \le \lambda(\alpha) + \lambda(\beta)$$

for all $\alpha, \beta > 0$, i.e. the mapping λ is subadditive on $(0,\infty)$.

(iii) If there exist $x_0, y_0 \in C$ with $f(x_0) > 0$ and $f(y_0) < 0$, then

$$\lambda(\alpha+\beta) = \lambda(\alpha) + \lambda(\beta)$$

for all $\alpha, \beta > 0$, i.e. the mapping λ is additive on $(0,\infty)$.

We have the following result providing many examples of subadditive functions $\lambda : [0,\infty) \to [0,\infty)$.

Theorem 2 ([26]) *Let* $h(z) = \sum_{n=0}^{\infty} a_n z^n$ *a power series with nonnegative coefficients* $a_n \geq 0$ *for all* $n \in \mathbb{N}$ *and convergent on the open disk* $D(0,R)$ *with* $R > 0$ *or* $R = \infty$. *If* $r \in (0,R)$ *then the function* $\lambda_r : [0,\infty) \to [0,\infty)$ *given by*

$$\lambda_r(t) := \ln\left[\frac{h(r)}{h(r\exp(-t))}\right] \tag{9}$$

is nonnegative, increasing and subadditive on $[0,\infty)$.

We have the following fundamental examples of power series with positive coefficients

$$h(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \ z \in D(0,1) \tag{10}$$

$$h(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = \exp(z) \qquad z \in \mathbb{C},$$

$$h(z) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n} = \cosh z, \ z \in \mathbb{C};$$

$$h(z) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1} = \sinh z, \ z \in \mathbb{C};$$

$$h(z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n = \ln \frac{1}{1-z}, \ z \in D(0,1).$$

Other important examples of functions as power series representations with positive coefficients are:

$$h(z) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1} = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right), \qquad z \in D(0,1); \tag{11}$$

$$h(z) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!} z^{2n+1} = \sin^{-1}(z), \qquad z \in D(0,1);$$

$$h(z) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1} = \tanh^{-1}(z), \qquad z \in D(0,1);$$

$$h(z) = {}_2F_1(\alpha,\beta,\gamma,z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} z^n, \alpha,\beta,\gamma > 0,$$

$$z \in D(0,1);$$

where Γ is *Gamma function.*

Remark 1 *Now, if we take* $h(z)=\frac{1}{1-z}$, $z\in D(0,1)$, *then*

$$\lambda_r(t)=\ln\left[\frac{1-r\exp(-t)}{1-r}\right] \tag{12}$$

is nonnegative, increasing and subadditive on $[0,\infty)$ *for any* $r\in(0,1)$.

If we take $h(z)=\exp(z)$, $z\in\mathbb{C}$ *then*

$$\lambda_r(t)=r[1-\exp(-t)] \tag{13}$$

is nonnegative, increasing and subadditive on $[0,\infty)$ *for any* $r>0$.

Corollary 1 ([26]) *Let* $h(z)=\sum_{n=0}^{\infty}a_nz^n$ *be a power series with nonnegative coefficients* $a_n\geq 0$ *for all* $n\in\mathbb{N}$ *and convergent on the open disk* $D(0,R)$ *with* $R>0$ *or* $R=\infty$ *and* $r\in(0,R)$. *For a mapping* $f:C\to\mathbb{R}$ *defined on convex subset* C *of a linear space* X, *the following statements are equivalent:*

(i) The function f *is* λ_r*-convex with* $\lambda_r:[0,\infty)\to[0,\infty)$,

$$\lambda_r(t):=\ln\left[\frac{h(r)}{h(r\exp(-t))}\right];$$

(ii) We have the inequality

$$\left[\frac{h(r)}{h(r\exp(-\alpha-\beta))}\right]^{f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)}\leq\left[\frac{h(r)}{h(r\exp(-\alpha))}\right]^{f(x)}\left[\frac{h(r)}{h(r\exp(-\beta))}\right]^{f(y)} \tag{14}$$

for any $\alpha,\beta\geq 0$ *with* $\alpha+\beta>0$ *and* $x,y\in C$.

(iii) We have the inequality

$$\frac{[h(r\exp(-\alpha))]^{f(x)}[h(r\exp(-\beta))]^{f(y)}}{[h(r\exp(-\alpha-\beta))]^{f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)}}\leq[h(r)]^{f(x)+f(y)-f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)} \tag{15}$$

for any $\alpha,\beta\geq 0$ *with* $\alpha+\beta>0$ *and* $x,y\in C$.

Remark 2 *We observe that, in the case when*

$$\lambda_r(t)=r[1-\exp(-t)],\ t\geq 0,$$

the function f *is* λ_r*-convex on convex subset* C *of a linear space* X *iff*

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)\leq\frac{[1-\exp(-\alpha)]f(x)+[1-\exp(-\beta)]f(y)}{1-\exp(-\alpha-\beta)} \tag{16}$$

for any $\alpha,\beta\geq 0$ *with* $\alpha+\beta>0$ *and* $x,y\in C$.

We observe that this definition is independent on $r>0$.

The inequality (16) is equivalent to

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)\leq\frac{\exp(\beta)\left[\exp(\alpha)-1\right]f(x)+\exp(\alpha)\left[\exp(\beta)-1\right]f(y)}{\exp(\alpha+\beta)-1} \tag{17}$$

for any $\alpha,\beta\geq 0$ *with* $\alpha+\beta>0$ *and* $x,y\in C$.

We have the following Jensen inequality for the Riemann integral [28]:

Theorem 3 *Let* $u:[a,b]\to[m,M]$ *be a Riemann integrable function on* $[a,b]$. *Let* $\lambda:[0,\infty)\to[0,\infty)$ *be a function with the property that* $\lambda(t)>0$ *for all* $t>0$ *and the function* $f:[m,M]\to[0,\infty)$ *is* λ*-convex and Riemann integrable on the interval* $[m,M]$. *If the following limit exists*

$$\lim_{t\to 0+}\frac{\lambda(t)}{t}=k\in(0,\infty) \tag{18}$$

then

$$f\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right)\leq\frac{k}{\lambda(b-a)}\int_a^b f(u(t))\,dt. \tag{19}$$

The following weighted version of Jensen inequality for the Riemann integral [28] also holds.

Theorem 4 *Let* $u,w:[a,b]\to[m,M]$ *be Riemann integrable functions on* $[a,b]$ *and* $w(t)\geq 0$ *for any* $t\in[a,b]$ *with* $\int_a^b w(t)\,dt>0$. *Let* $\lambda:[0,\infty)\to[0,\infty)$ *be a function with the property that* $\lambda(t)>0$ *for all* $t>0$ *and the function* $f:[m,M]\to[0,\infty)$ *is* λ*-convex and Riemann integrable on the interval* $[m,M]$. *If the following limit exists, is finite and*

$$\lim_{t\to\infty}\frac{t}{\lambda(t)}=\ell>0, \tag{20}$$

then

$$f\left(\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\,u(t)\,dt\right)\leq\ell\frac{1}{\int_a^b w(t)\,dt}\int_a^b\lambda(w(t))\,f(u(t))\,dt. \tag{21}$$

Motivated by the above results in this paper we establish some Jensen type inequalities for the general Lebesgue integral.

1 Some Results for Differentiable Functions

If we assume that the function $f : I \to [0, \infty)$ is differentiable on the interior of I, denoted by $\mathring{I}$, then we have the following *"gradient inequality"* that will play an essential role in the following.

Lemma 1 *Let $\lambda : (0, \infty) \to (0, \infty)$ be a function such that the right limit*

$$\lim_{t \to 0+} \frac{\lambda(t)}{t} = k \in (0, \infty) \tag{22}$$

exists and is finite, and the left derivative in 1 denoted by $\lambda'_{-}(1)$ exists and is finite.

If the function $f : I \to [0, \infty)$ is differentiable on $\mathring{I}$ and λ-convex, then

$$kf(x) - \lambda'_{-}(1) f(y) \geq \lambda(1) f'(y)(x - y) \tag{23}$$

for any $x, y \in \mathring{I}$ with $x \neq y$.

Proof. Since f is λ-convex on I, then

$$\frac{\lambda(t) f(x) + \lambda(1-t) f(y)}{\lambda(1)} \geq f(tx + (1-t) y)$$

for any $t \in (0, 1)$ and for any $x, y \in \mathring{I}$, which is equivalent to

$$\lambda(t) f(x) + [\lambda(1-t) - \lambda(1)] f(y) \geq \lambda(1) [f(tx + (1-t) y) - f(y)]$$

and by dividing by $t > 0$ we get

$$\frac{\lambda(t)}{t} f(x) + \left[\frac{\lambda(1-t) - \lambda(1)}{t}\right] f(y) \geq \lambda(1) \frac{f(tx + (1-t) y) - f(y)}{t} \tag{24}$$

for any $t \in (0, 1)$.

Now, since f is differentiable on $y \in \mathring{I}$, then we have

$$\begin{aligned} \lim_{t \to 0+} \frac{f(tx + (1-t) y) - f(y)}{t} &= \lim_{t \to 0+} \frac{f(y + t(x-y)) - f(y)}{t} \\ &= (x - y) \lim_{t \to 0+} \frac{f(y + t(x-y)) - f(y)}{t(x-y)} \\ &= (x - y) f'(y) \end{aligned} \tag{25}$$

for any $x \in \mathring{I}$ with $x \neq y$.

Also we have

$$\begin{aligned} \lim_{t \to 0+} \frac{\lambda(1-t) - \lambda(1)}{t} &= \lim_{s \to 1-} \frac{\lambda(s) - \lambda(1)}{1 - s} \\ &= -\lim_{s \to 1-} \frac{\lambda(s) - \lambda(1)}{s - 1} = -\lambda'_{-}(1) \end{aligned} \tag{26}$$

Taking the limit over $t \to 0+$ in (24) and utilizing (25) and (26) we get the desired result (23). $\square$

Remark 3 *If we assume that*

$$k \geq \lambda'_{-}(1), \tag{27}$$

then the inequality (23) also holds for $x = y$.

Remark 4 *If* $\lambda : [0, \infty) \to [0, \infty)$ *with* $\lambda(0) = 0$ *then the condition (22) is equivalent to the fact that the right derivative*

$$\lambda'_{+}(0) = \lim_{t \to 0+} \frac{\lambda(t)}{t}$$

exists, is finite and $\lambda'_{+}(0) = k$.

In this situation the inequality (23) becomes for $\lambda'_{+}(0) > 0$

$$\lambda'_{+}(0) f(x) - \lambda'_{-}(1) f(y) \geq \lambda(1) f'(y)(x - y) \tag{28}$$

for any $x, y \in \mathring{I}$ *with* $x \neq y$.

If the function λ *is subadditive on* $[0, \infty)$ *and has finite lateral derivatives with* $\lambda'_{+}(0) > 0$, *then*

$$\lambda(t) + \lambda(1 - t) \geq \lambda(1), \ t \in (0, 1),$$

i.e.

$$\frac{\lambda(t)}{t} \geq \frac{\lambda(1) - \lambda(1 - t)}{t}, \ t \in (0, 1). \tag{29}$$

Taking the limit over $t \to 0+$ *in (29) we get*

$$\lambda'_{+}(0) \geq \lambda'_{-}(1),$$

therefore the inequality (28) also holds for $x = y$.

We have the following result.

Corollary 2 *Let* $\lambda : [0, \infty) \to [0, \infty)$ *be a subadditive function with* $\lambda(0) = 0$ *and having the lateral derivative* $\lambda'_{+}(0), \lambda'_{-}(1) \in (0, \infty)$.

If the function $f : I \to [0, \infty)$ *is differentiable on* $\mathring{I}$ *and* λ*-convex, then*

$$\lambda'_{+}(0) f(x) - \lambda'_{-}(1) f(y) \geq \lambda(1) f'(y)(x - y) \tag{30}$$

for any $x, y \in \mathring{I}$.

As examples of such functions we have:

Proposition 2 *Let $h(z)=\sum_{n=0}^{\infty}a_n z^n$ a power series with nonnegative coefficients $a_n\geq 0$ for all $n\in\mathbb{N}$ and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$ and $r\in(0,R)$. If the function $f:I\to[0,\infty)$ is differentiable on $\mathring{I}$ and λ_r-convex with $\lambda_r:[0,\infty)\to[0,\infty)$,*

$$\lambda_r(t):=\ln\left[\frac{h(r)}{h(r\exp(-t))}\right],$$

then

$$\frac{rh'(r)}{h(r)}f(x)-\frac{re^{-1}h'(re^{-1})}{h(re^{-1})}f(y)\geq\ln\left[\frac{h(r)}{h(re^{-1})}\right]f'(y)(x-y), \tag{31}$$

for any $x,y\in\mathring{I}$.

Proof. We know that λ_r is differentiable on $(0,\infty)$ and

$$\lambda_r'(t):=\frac{r\exp(-t)\,h'(r\exp(-t))}{h(r\exp(-t))}$$

for $t\in(0,\infty)$, where

$$h'(z)=\sum_{n=1}^{\infty}na_n z^{n-1}.$$

Since $\lambda_r(0)=0$, then

$$k=\lim_{s\to 0+}\frac{\lambda(s)}{s}=\lambda_+'(0)=\frac{rh'(r)}{h(r)}>0\text{ for }r\in(0,R).$$

Also

$$\lambda_r'(1)=\frac{re^{-1}h'(re^{-1})}{h(re^{-1})}$$

and

$$\lambda_r(1)=\ln\left[\frac{h(r)}{h(re^{-1})}\right].$$

Applying Corollary 2 we deduce the desired result (31). □

Corollary 3 *If the function $f:I\to[0,\infty)$ is differentiable on $\mathring{I}$ and λ-convex with $\lambda:[0,\infty)\to[0,\infty)$, $\lambda(t)=1-\exp(-t)$, then we have*

$$ef(x)-f(y)\geq(e-1)f'(y)(x-y) \tag{32}$$

for any $x,y\in\mathring{I}$.

It follows by Proposition 2 observing that $\lambda'(t)=\exp(-t)$, $t>0$.

2 Jensen Type Inequalities

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space consisting of a set Ω, a σ −algebra $\mathcal{A}$ of parts of Ω and a countably additive positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. For a $\mu-$measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ − a.e.(almost every) $x \in \Omega$, consider the Lebesgue space

$$L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega w(x) |f(x)| \, d\mu(x) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x) \, d\mu(x)$.

Theorem 5 *Let $\lambda : [0, \infty) \to [0, \infty)$ be a subadditive function with $\lambda(0) = 0$ and having the lateral derivative $\lambda'_+(0)$, $\lambda'_-(1) \in (0, \infty)$. If the function $f : I \to [0, \infty)$ is differentiable on $\mathring{I}$ and λ-convex, then for any $u : \Omega \to [m, M] \subset \mathring{I}$ so that $f \circ u$, $u \in L_w(\Omega, \mu)$, where $w \geq 0$ μ-a.e. (μ-almost everywhere) on Ω with $\int_\Omega w d\mu = 1$ we have*

$$\int_\Omega w \cdot (f \circ u) \, d\mu \geq \frac{\lambda'_-(1)}{\lambda'_+(0)} f\left(\int_\Omega wu d\mu\right). \tag{33}$$

Proof. Observe that, since $u : \Omega \to [m, M]$ and $u \in L_w(\Omega, \mu)$, then $\int_\Omega wu d\mu \in [m, M]$. Applying Corollary 2 we have

$$\lambda'_+(0) f(u(t)) - \lambda'_-(1) f\left(\int_\Omega wu d\mu\right) \tag{34}$$
$$\geq \lambda(1) f'\left(\int_\Omega wu d\mu\right)\left(u(t) - \int_\Omega wu d\mu\right)$$

for any $t \in \Omega$.

Multiplying (34) by $w(t) \geq 0$ for μ-almost every $t \in \Omega$ we get

$$\lambda'_+(0) w(t) f(u(t)) - \lambda'_-(1) f\left(\int_\Omega wu d\mu\right) w(t) \tag{35}$$
$$\geq \lambda(1) f'\left(\int_\Omega wu d\mu\right)\left(w(t) u(t) - \left(\int_\Omega wu d\mu\right) w(t)\right)$$

for μ-almost every $t \in \Omega$.

Integrating (35) over t on Ω we get

$$\lambda'_+(0) \int_\Omega w(t) f(u(t)) \, d\mu(t) - \lambda'_-(1) f\left(\int_\Omega wu d\mu\right) \int_\Omega w(t) \, d\mu(t) \tag{36}$$
$$\geq \lambda(1) f'\left(\int_\Omega wu d\mu\right)$$
$$\times \left(\int_\Omega w(t) u(t) \, d\mu(t) - \left(\int_\Omega wu d\mu\right) \int_\Omega w(t) \, d\mu(t)\right)$$

and since $\int_\Omega w(t) \, d\mu(t) = 1$, we deduce the desired result (33). □

The following inequality of Hermite-Hadamard type holds:

Corollary 4 *Let $\lambda : [0, \infty) \to [0, \infty)$ be a subadditive function with $\lambda(0) = 0$ and having the lateral derivative $\lambda'_+(0), \lambda'_-(1) \in (0, \infty)$. If the function $f : I \to [0, \infty)$ is differentiable on $\mathring{I}$ and λ-convex, then for any $[a, b] \subset \mathring{I}$ we have*

$$\frac{1}{b-a} \int_a^b f(t)\, dt \geq \frac{\lambda'_-(1)}{\lambda'_+(0)} f\left(\frac{a+b}{2}\right). \tag{37}$$

It follows from Theorem 5 by taking $\Omega = [a, b]$, $u : [a, b] \to [a, b]$, $u(t) = t$, $w(t) = \frac{1}{b-a}$ and $d\mu = dt$ being the Lebesgue measure on the interval $[a, b]$.

The inequality (37) provides other lower bound for the integral mean than the first inequality in (7). Since for h-convexity, $h(0)$ may not be defined, the lower bounds from (37) and (7) cannot be compared in general.

If we consider the discrete measure, then we have:

Corollary 5 *Let $\lambda : [0, \infty) \to [0, \infty)$ be a subadditive function with $\lambda(0) = 0$ and having the lateral derivative $\lambda'_+(0), \lambda'_-(1) \in (0, \infty)$. If the function $f : I \to [0, \infty)$ is differentiable on $\mathring{I}$ and λ-convex, then for any $x_i \in \mathring{I}$ and $p_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^n p_i = 1$ we have*

$$\sum_{i=1}^n p_i f(x_i) \geq \frac{\lambda'_-(1)}{\lambda'_+(0)} f\left(\sum_{i=1}^n p_i x_i\right).$$

Remark 5 *Let $h(z) = \sum_{n=0}^\infty a_n z^n$ be a power series with nonnegative coefficients $a_n \geq 0$ for all $n \in \mathbb{N}$ and convergent on the open disk $D(0, R)$ with $R > 0$ or $R = \infty$ and $r \in (0, R)$. Assume that the function $f : I \to [0, \infty)$ is differentiable on $\mathring{I}$ and λ_r-convex with $\lambda_r : [0, \infty) \to [0, \infty)$,*

$$\lambda_r(t) := \ln\left[\frac{h(r)}{h(r\exp(-t))}\right].$$

If $f : I \to [0, \infty)$ is differentiable on $\mathring{I}$ and λ_r-convex, then for any $u : \Omega \to [m, M] \subset \mathring{I}$ so that $f \circ u$, $u \in L_w(\Omega, \mu)$, where $w \geq 0$ μ-a.e. (μ-almost everywhere) on Ω with $\int_\Omega w d\mu = 1$ we have

$$\int_\Omega w \cdot (f \circ u)\, d\mu \geq \frac{e^{-1} h'(re^{-1})\, h(r)}{h(re^{-1})\, h'(r)} f\left(\int_\Omega wu d\mu\right). \tag{38}$$

Remark 6 *If the function $f : I \to [0, \infty)$ is differentiable on $\mathring{I}$ and λ-convex with $\lambda : [0, \infty) \to [0, \infty)$, $\lambda(t) = 1 - \exp(-t)$, then for any $[a, b] \subset \mathring{I}$ we have*

$$\frac{1}{b-a} \int_a^b f(t)\, dt \geq \frac{1}{e} f\left(\frac{a+b}{2}\right). \tag{39}$$

Also, for any $x_i \in \mathring{I}$ *and* $p_i \geq 0$, $i \in \{1, ..., n\}$ *with* $\sum_{i=1}^{n} p_i = 1$ *we have*

$$\sum_{i=1}^{n} p_i f(x_i) \geq \frac{1}{e} f\left(\sum_{i=1}^{n} p_i x_i\right). \tag{40}$$

Recall Slater's inequality for differentiable convex functions [56]:

Lemma 2 *Let* $f : I \to \mathbb{R}$ *be a nondecreasing (nonincreasing) differentiable convex function on* I, $x_i \in I$, $p_i \geq 0$ *with* $P_n = \sum_{i=1}^{n} p_i > 0$ *and assume that* $\sum_{i=1}^{n} p_i f'(x_i) \neq 0$. *Then one has the inequality*

$$f\left(\frac{\sum_{i=1}^{n} p_i x_i f'(x_i)}{\sum_{i=1}^{n} p_i f'(x_i)}\right) \geq \frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i). \tag{41}$$

As shown in [22, pp. 129-130], the monotonicity condition in Lemma 2 can be weakened by assuming that

$$\frac{\sum_{i=1}^{n} p_i x_i f'(x_i)}{\sum_{i=1}^{n} p_i f'(x_i)} \in I.$$

We can state the following result that is similar to Slater's inequality:

Theorem 6 *Let* $\lambda : [0, \infty) \to [0, \infty)$ *be a subadditive function with* $\lambda(0) = 0$ *and having the lateral derivative* $\lambda'_+(0)$, $\lambda'_-(1) \in (0, \infty)$. *If the function* $f : I \to [0, \infty)$ *is differentiable on* $\mathring{I}$ *and* λ*-convex, then for any* $u : \Omega \to [m, M] \subset \mathring{I}$ *so that* $f \circ u$, $u \cdot (f' \circ u)$, $f' \circ u \in L_w(\Omega, \mu)$, *where* $w \geq 0$ μ*-a.e. (*μ*-almost everywhere) on* Ω *with* $\int_\Omega w d\mu = 1$ *and*

$$\frac{\int_\Omega wu \cdot (f' \circ u) d\mu}{\int_\Omega w \cdot (f' \circ u) d\mu} \in [m, M],$$

we have

$$\frac{\lambda'_+(0)}{\lambda'_-(1)} f\left(\frac{\int_\Omega wu \cdot (f' \circ u) d\mu}{\int_\Omega w \cdot (f' \circ u) d\mu}\right) \geq \int_\Omega w \cdot (f \circ u) d\mu. \tag{42}$$

Proof. Since the function $f : I \to [0, \infty)$ is differentiable on $\mathring{I}$ and λ-convex, then by (30) we have

$$\lambda'_+(0) f(x) - \lambda'_-(1) f(u(t)) \geq \lambda(1) f'(u(t))(x - u(t)) \tag{43}$$

for any $x \in \mathring{I}$ and $t \in \Omega$.

If we multiply by $w(t) \geq 0$ and integrate we get

$$\begin{aligned} &\lambda'_+(0) f(x) - \lambda'_-(1) \int_\Omega w(t) f(u(t)) d\mu(t) \\ &\geq \lambda(1) x \int_\Omega w(t) f'(u(t)) d\mu(t) - \int_\Omega w(t) f'(u(t)) u(t) d\mu(t), \end{aligned} \tag{44}$$

for any $x \in \mathring{I}$.

Since $\int_\Omega w(t) f'(u(t)) d\mu(t) \neq 0$ and

$$x_0 := \frac{\int_\Omega w(t) f'(u(t)) u(t) d\mu(t)}{\int_\Omega w(t) f'(u(t)) d\mu(t)} \in [m, M],$$

then by taking $x = x_0$ in (44) we get the desired result (42). □

The following Hermite-Hadamard type inequality holds:

Corollary 6 *Let $\lambda : [0,\infty) \to [0,\infty)$ be a subadditive function with $\lambda(0) = 0$ having the lateral derivative $\lambda'_+(0), \lambda'_-(1) \in (0,\infty)$. If the function $f : I \to [0,\infty)$ is differentiable on $\mathring{I}$ and λ-convex, and for $[a,b] \subset \mathring{I}$ we have*

$$\frac{\int_a^b t f'(t) dt}{f(b) - f(a)} = \frac{b f(b) - a f(a) - \int_a^b f(t) dt}{f(b) - f(a)} \in [a,b], \tag{45}$$

then we have

$$\frac{\lambda'_+(0)}{\lambda'_-(1)} f\left(\frac{b f(b) - a f(a) - \int_a^b f(t) dt}{f(b) - f(a)}\right) \geq \frac{1}{b-a} \int_a^b f(t) dt. \tag{46}$$

The following discrete inequality also holds:

Corollary 7 *Let $\lambda : [0,\infty) \to [0,\infty)$ be a subadditive function with $\lambda(0) = 0$ having the lateral derivative $\lambda'_+(0), \lambda'_-(1) \in (0,\infty)$. If the function $f : I \to [0,\infty)$ is differentiable on $\mathring{I}$ and λ-convex, then for any $x_i \in \mathring{I}$ and $p_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^n p_i = 1$ and*

$$\frac{\sum_{i=1}^n p_i x_i f'(x_i)}{\sum_{i=1}^n p_i f'(x_i)} \in \mathring{I},$$

we have

$$\frac{\lambda'_+(0)}{\lambda'_-(1)} f\left(\frac{\sum_{i=1}^n p_i x_i f'(x_i)}{\sum_{i=1}^n p_i f'(x_i)}\right) \geq \sum_{i=1}^n p_i f(x_i). \tag{47}$$

Remark 7 *The interested reader can obtain some particular inequalities of interest by taking λ_r-convex functions with $\lambda_r : [0,\infty) \to [0,\infty)$,*

$$\lambda_r(t) := \ln\left[\frac{h(r)}{h(r\exp(-t))}\right],$$

and h is as in Theorem 2. The details are omitted.

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