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An iterative algorithm based on the generalized viscosity explicit methods for an infinite family of accretive operators

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Abstract. In this paper, we introduce and study a new iterative method based on the generalized viscosity explicit methods (GVEM) for solving the inclusion problem with an infinite family of multivalued accretive operators in real Banach spaces. Applications to equilibrium and to convex minimization problems involving an infinite family of semi-continuous and convex functions are included. Our results improve important recent results.

Key Words: Proximal-point algorithm, Generalized viscosity explicit methods, Accretive operators, Common zeros
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Introduction

Let H be a real Hilbert space. For a multivalued map $A : H \to 2^H$, the domain of A, $D(A)$, the image of a subset S of H, $A(S)$, the range of A, $R(A)$, and the graph of A, $G(A)$, are defined as follows:

$$D(A) := \{x \in H : Ax \neq \emptyset\}, \qquad A(S) := \cup\{Ax : x \in S\},$$

$$R(A) := A(H), \qquad G(A) := \{[x, u] : x \in D(A), u \in Ax\}.$$

A multivalued map $A : D(A) \subset H \to 2^H$ is called monotone if the inequality

$$\langle u - v, x - y\rangle \geq 0$$

holds for each $x, y \in D(A)$, $u \in Ax$, $v \in Ay$. A monotone operator A is called maximal monotone if its graph $G(A)$ is not properly contained in the graph of any other monotone operator. It is well known that A is maximal monotone if and only if A is monotone and $R(I + rA) = H$ for all $r > 0$ and A is said to satisfy the range condition if $\overline{D(A)} \subset R(I + rA)$. Many problems arising in different areas of mathematics, such as optimization, variational analysis, and differential equations, can be modeled by the equation

$$0 \in Ax, \tag{1}$$

where A is a monotone mapping. The solution set of this equation coincides with the null points set of A. Such operators have been studied extensively (see, e.g., Bruck Jr [6], Rockafellar [25], Xu [26], and the references therein). In fact, $f : H \to \mathbb{R} \cup \{\infty\}$ be a proper lower semicontinuous and convex function, then, it is known that the multivalued map $A := \partial f$, the subdifferential of f, is maximal monotone (see, e.g., [19]). For $w \in H$ from $w \in \partial f(x)$ it follows that $f(y) - f(x) \geq \langle y - x, w \rangle$ for all $y \in H$, and hence, $x \in \operatorname{Argmin}\Big(f - \langle \cdot, w \rangle\Big)$. In particular, the inclusion (1) is equivalent to

$$f(x) = \min_{y \in H} f(y).$$

The problem (1) has been studied by numerous researchers. A popular method used to solve (1) by iterations is the proximal-point algorithm proposed by Rockafellar [25], which is recognized as a powerful and successful algorithm for finding zeros of monotone operators.

An early fundamental result in the theory of accretive operators, due to Browder [3], states that the initial value problem of ordinary differential equation

$$\frac{du}{dt} + Au = 0, \;\; u(0) = u_0, \tag{2}$$

is solvable when A is locally lipshitzian and monotone on H. For obtaining the numerical solution of Eq. (2), numerous authors devoted themselves to probing methods of approximating and harvested fruitful results. One of the powerful numerical methods for the numerical solution of equation Eq. (2) is the implicit midpoint rule (IMR) which generates an iterative sequence $\{x_n\}$ via the relation

$$\frac{1}{h}(x_{n+1} - x_n) = A\Big(\frac{x_{n+1} - x_n}{2}\Big). \tag{3}$$

The sequence $\{x_n\}$ generated by (3) converges to the exact solution of (2) (see e.g., Auzinger [1], Bader[2]). In recent decades, a host of mathematicians concentrated on approximating the solution of (1) (when it exists) which coincides with the equilibrium state $\frac{du}{dt} = 0$, $Au = 0$, because a variety of problems, for example, convex optimization, linear programming, monotone inclusions, and elliptic differential equations can be formulated as the equilibrium state. Therefore, finding a zero of nonlinear operator A is an important task in approximation theory. In studying the inclusion problem (1) where A is a monotone operator, Browder [4] introduced an operator $T : H \to H$ by $T = I - A$ where I is the identity mapping on H. The operator T is nonexpansive, and the zeros of A, if they exist, correspond to the fixed points of T. Thus, approximating the solution of (1) is transferred to approximating the fixed points of nonexpansive mappings.

For nonexpansive mappings with fixed points, Mann iterative method [17] is a valuable tool for studying them. However, only weak convergence is guaranteed in infinite dimensional spaces. Thus, a natural question arises: could we obtain a strong convergence theorem by using the well-known Krasnoselskii-Mann method for non-expansive mappings? In this connection, in 1975, Genel and Lindenstrauss [10] gave a counterexample. Hence, the modification is necessary in order to guarantee the strong convergence of Krasnoselskii-Mann method. Therefore, many authors try to modify Mann's iteration to have strong convergence for nonlinear operators.

Recently, an iterative sequence for the explicit midpoint rule has been studied by many authors, because it is a powerful method for solving ordinary differential equations; see, for example, [28, 16, 13, 18] and the references therein.

In 2017, Marino et al. [18], motivated by the fact that explicit midpoint rule is remarkably useful for finding fixed points of single-valued nonexpansive mapping, proved the following theorem.

Theorem 1 *Let H be a real Hilber space and K, a closed and convex subset of H. Let $T : C \to C$ be a quasi-nonexpansive mapping and $f : C \to C$ be a contraction. Assume that $I - T$ is demiclosed in 0 and $F(T) = \{x \in C : Tx = x\} \neq \emptyset$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in C$ by*

$$\begin{cases} \bar{x}_{n+1} = \beta_n x_n + (1 - \beta_n)Tx_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T\big(s_n x_n + (1 - s_n)\bar{x}_{n+1}\big), \end{cases} \tag{4}$$

where $\{s_n\}$, $\{\beta_n\}$ and $\{\alpha_n\}$ are sequences in $(0, 1)$ satisfying

$$(i)\ \lim_{n\to\infty} \alpha_n = 0; \qquad (ii)\ \sum_{n=0}^{\infty} \alpha_n = \infty,\ \limsup_n \beta_n(1 - \beta_n)(1 - s_n) > 0.$$

Then, the sequence $\{x_n\}$ generated by (4) converges strongly to $x^* \in F(T)$ that is the unique solution in $F(T)$ of the variational inequality

$$\langle x^* - f(x^*), x^* - p\rangle \leq 0 \quad \text{for all } p \in F(T). \tag{5}$$

In the present paper, we introduce a new iterative algorithm which is a combination of viscosity approximation method and a modified Mann algorithm for approximating common zeros for a countable infinite family of accretive operators which is also a solution of some variational inequality problems in real Banach spaces. Finally, we apply our main result to equilibrium and to convex minimization problems.

1 Preliminairies

Let E be a Banach space with norm $\|\cdot\|$ and dual E^*. For any $x \in E$ and $x^* \in E^*$, $\langle x^*, x\rangle$ is used to refer to $x^*(x)$. Let $\varphi : [0, +\infty) \to [0, \infty)$ be a stricly increasing continuous function such that $\varphi(0) = 0$ and $\varphi(t) \to +\infty$ as $t \to \infty$. Such a function φ is called gauge. Associated to a gauge φ a duality map $J_\varphi : E \to 2^{E^*}$ is defined by:

$$J_\varphi(x) := \{x^* \in E^* : \langle x, x^*\rangle = ||x||\varphi(||x||), ||x^*|| = \varphi(||x||)\}, \quad x \in E. \quad (6)$$

If the gauge is defined by $\varphi(t) = t$, then the corresponding duality map is called the *normalized duality map* and is denoted by J. Hence, the normalized duality map is given by

$$J(x) := \{x^* \in E^* : \langle x, x^*\rangle = ||x||^2 = ||x^*||^2 =\} \text{ for all } x \in E.$$

Notice that

$$J_\varphi(x) = \frac{\varphi(||x||)}{||x||} J(x), \quad x \neq 0.$$

A normed linear space E is said to be strictly convex if the following holds: from $\|x\| = \|y\| = 1$ and $x \neq y$ it follows that $\left\|\frac{x+y}{2}\right\| < 1$. The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) := \inf\left\{1 - \frac{1}{2}\|x + y\| \ : \|x\| = \|y\| = 1, \ \|x - y\| \geq \epsilon\right\}.$$

E is *uniformly convex* if and only if $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0, 2]$. For $p > 1$, E is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$ for all $\epsilon \in (0, 2]$.

Let E be a real normed space and let $S := \{x \in E : \|x\| = 1\}$. E is said to be *smooth* if the limit

$$\lim_{t\to 0^+} \frac{\|x + ty\| - \|x\|}{t} \quad (7)$$

exists for each $x, y \in S$. E is said to be *uniformly smooth* if it is smooth and the convergence in (7) is uniform for each $x, y \in S$.

Let J_q denote the *generalized duality mapping* from E to 2^{E^*} defined by

$$J_q(x) := \left\{f \in E^* : \langle x, f\rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1}\right\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. J_2 is called the *normalized duality mapping* and is denoted by J.

It is known that E is smooth if and only if each duality map J_φ is single-valued, that E is Frechet differentiable if and only if each duality map J_φ is norm-to-norm continuous in E, and that E is uniformly smooth if and only if each duality map J_φ is norm-to-norm uniformly continuous on bounded subsets of E.

Following Browder [4], we say that a Banach space has a weakly continuous duality map if there exists a gauge φ such that J_φ is a single-valued and is weak-to-weak* sequentially continuous, i.e., from $(x_n) \subset E$ and $x_n \xrightarrow{w} x$ it follows that $J_\varphi(x_n) \xrightarrow{w^*} J_\varphi(x))$. It is known that l^p $(1 < p < \infty)$ has a weakly continuous duality map with gauge $\varphi(t) = t^{p-1}$ (see [8] fore more details on duality maps).

Finally, recall that a Banach space E satisfies Opial's property (see, e.g., [21]) if $\limsup\limits_{n\to+\infty} \|x_n - x\| < \limsup\limits_{n\to+\infty} \|x_n - y\|$ whenever $x_n \xrightarrow{w} x$, $x \neq y$. A Banach space E that has a weakly continuous duality map satisfies Opial's property. Given a gauge φ and a smooth real Banach space E, the map $A : D(A) \subset E \to 2^E$ is called *accretive* if

$$\langle u - v, J_\varphi(x - y)\rangle \geq 0, \qquad (x, u), (y, v) \in G(A).$$

A multi-valued map A defined on a real Banach space E is called m-accretive if it is accretive and $R(I + rA) = E$ for some $r > 0$, and it is said to satisfy the range condition if $R(I + rA) = E$ for all $r > 0$.

Example 1 *Let $A : \mathbb{R} \to 2^{\mathbb{R}}$ be a map defined by*

$$Ax = \begin{cases} sgn(x), & x \neq 0, \\ [-1, 1], & x = 0, \end{cases} \tag{8}$$

where A is the subdifferential of the absolute value function, $\partial|\cdot|$. Then A is m-accretive. It can be shown that if $R(I + rA) = E$ for some $r > 0$, then this holds for all $s > 0$. Hence, the m-accretive condition implies the range condition.

Definition 1 *Let E be a real Banach space and $T : D(T) \subset E \to E$ be a mapping. The mapping $I - T$ is said to be demiclosed at* 0 *if for any sequence $\{x_n\} \subset D(T)$ such that $\{x_n\}$ converges weakly to p and $\|x_n - Tx_n\| \to 0$ one has $p \in F(T)$.*

Lemma 1 ([4]) *Let E be a Banach space satisfying Opial's property, K be a closed convex subset of E, and $T : K \to K$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Then $I - T$ is demiclosed; that is,*

$$\{x_n\} \subset K, \quad x_n \rightharpoonup x \in K \quad \textit{and} \quad (I - T)x_n \to y \textit{ implies } (I - T)x = y.$$

Lemma 2 ([15]) *Assume that a Banach space E has a weakly continuous duality mapping J_φ with gauge φ. Then, for all $x, y \in E$,*

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x+y)\rangle, \tag{9}$$

where $\Phi(t) = \displaystyle\int_0^t \varphi(\sigma)d\sigma,\ t \geq 0$. In particular, for the normalized duality mapping, one has the important special version of (9),

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y)\rangle,$$

for all $x, y \in E$.

Lemma 3 ([27]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1-\alpha_n)a_n + \sigma_n$ for all $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that*

$$(a) \sum_{n=0}^{\infty} \alpha_n = \infty, \qquad (b) \limsup_{n\to\infty} \frac{\sigma_n}{\alpha_n} \leq 0 \quad \text{or} \quad \sum_{n=0}^{\infty} |\sigma_n| < \infty.$$

Then $\lim\limits_{n\to\infty} a_n = 0$.

Lemma 4 ([9]) *Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B(0)_r := \{x \in E : ||x|| \leq r\}$ be a closed ball with center 0 and radius r. Then for any given sequence $\{u_1, u_2, ..., u_n, ...\} \subset B(0)_r$ and any positive real numbers $\{\lambda_1, \lambda_2, ..., \lambda_n, ...\}$ with $\sum\limits_{k=1}^{\infty} \lambda_k = 1$, there exists a continuous, strictly increasing and convex function $g : [0, 2r] \to \mathbb{R}^+$, $g(0) = 0$, such that for any integer i, j with $i < j$,*

$$\Big\|\sum_{k=1}^{\infty} \lambda_k u_k\Big\|^2 \leq \sum_{k=1}^{\infty} \lambda_k \|u_k\|^2 - \lambda_i\lambda_j g(\|u_i - u_j\|).$$

Let C be a nonempty subset of a real Banach space E. A mapping $Q_C : E \to C$ is said to be sunny if

$$Q_C(Q_Cx + t(x - Q_Cx)) = Q_Cx$$

for each $x \in E$ and $t \geq 0$. A mapping $Q_C : E \to C$ is said to be a retraction if $Q_Cx = x$ for each $x \in C$.

Lemma 5 ([23]) *Let C and D be nonempty subsets of a smooth real Banach space E with $D \subset C$ and let $Q_D : C \to D$ be a retraction from C into D. Then Q_D is sunny and nonexpansive if and only if*

$$\langle z - Q_Dz, J(y - Q_Dz)\rangle \leq 0 \tag{10}$$

for all $z \in C$ and $y \in D$.

Note that Lemma 5 still holds if the normalized duality map is replaced by the general duality map J_φ.

Remark 1 *If K is a nonempty closed convex subset of a Hilbert space H, then the nearest point projection P_K from H to K is the sunny nonexpansive retraction.*

The resolvent operator has the following properties:

Lemma 6 ([11]) *Let E be a real Banach space and let $A : D(A) \subset E \to E$ be a set-valued operator. For any $r > 0$,*
(i) A is accretive if and only if the resolvent J_r^A of A is single-valued and nonexpansive;
(ii) A is m-accretive if and only if J_r^A of A is single-valued and nonexpansive and its domain is the entire E;
(iii) $0 \in A(x^*)$ if and only if $x^* \in F(J_r^A)$, where $F(J_r^A)$ denotes the fixed-point set of J_r^A.

Lemma 7 ([20]) *For any $r > 0$ and $\mu > 0$, the following holds:*

$$\frac{\mu}{r}x + \left(1 - \frac{\mu}{r}\right) J_r^A x \in D(J_r^A)$$

and

$$J_r^A x = J_\mu^A \left(\frac{\mu}{r}x + \left(1 - \frac{\mu}{r}\right) J_r^A x\right).$$

Lemma 8 ([8]) *Let A be a continuous accretive operator mapping defined on a real Banach space E with $D(A) = E$. Then A is m-accretive.*

2 Main results

We now formulate our main result.

Theorem 2 *Let E be a uniformly convex real Banach space having a weakly continuous duality map J_φ. Let K be a nonempty, closed and convex subset of E and $f : K \to K$ be contraction mapping with a constant $b \in [0, 1)$. Let B_i, $i \in \mathbb{N}^*$ be an infinite family of multivalued accretive operators of E such that $F := \bigcap_{i=1}^{\infty} B_i^{-1}(0) \neq \emptyset$ and $\bigcap_{i=1}^{\infty} \overline{D(B_i)} \subset K \subset \bigcap_{i=1}^{\infty} R(I + rB_i)$ for all $r > 0$. Let $\{\beta_{n,i}\}$, $\{s_n\}$, $\{\alpha_n\}$ be three sequences in $(0, 1)$ and $\{r_n\} \subset]0, \infty[$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in K$ by:*

$$\begin{cases} \bar{x}_{n+1} = \beta_{n,0}x_n + \displaystyle\sum_{i=1}^{\infty} \beta_{n,i} J_{r_n}^{B_i} x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)\big(s_n x_n + (1 - s_n)\bar{x}_{n+1}\big), \end{cases} \tag{11}$$

Suppose the following conditions hold:

$$(i)\ \lim_{n\to\infty}\alpha_n = 0; \qquad (ii)\ \sum_{n=0}^{\infty}\alpha_n = \infty, \quad \sum_{i=0}^{\infty}\beta_{n,i} = 1,$$

$$(iii)\ \liminf_{n\to\infty} r_n > 0 \text{ and } \liminf_{n\to\infty}(1-s_n)\beta_{n,0}\beta_{n,i} > 0 \text{ for all } i \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ *generated by* (11) *converges strongly to* $x^* \in F$, *which is a unique solution of variational inequality*

$$\langle x^* - f(x^*), J_\varphi(x^* - p)\rangle \le 0, \qquad p \in F. \tag{12}$$

Proof. First we show the uniqueness of a solution of the variational inequality (12). Suppose both $x^* \in F$ and $x^{**} \in F$ are solutions to (12), then

$$\langle x^* - f(x^*), J_\varphi(x^* - x^{**})\rangle \le 0 \tag{13}$$

and

$$\langle x^{**} - f(x^{**}), J_\varphi(x^{**} - x^*)\rangle \le 0. \tag{14}$$

Adding up (13) and (14), we obtain

$$\langle x^{**} - x^* + f(x^*) - f(x^{**}), J_\varphi(x^{**} - x^*)\rangle \le 0. \tag{15}$$

Further,

$$\langle x^{**} - x^* + f(x^*) - f(x^{**}), J_\varphi(x^{**} - x^*)\rangle \ge (1-b)\varphi(\|x^* - x^{**}\|)\|x^* - x^{**}\|,$$

which implies that $x^* = x^{**}$, and the uniqueness is proved. Below we use x^* to denote the unique solution of (12).

For each $n \ge 1$, put $z_n := s_n x_n + (1-s_n)\bar{x}_{n+1}$, and let $p \in F$. We have

$$\begin{aligned}
\|z_n - p\| &= \|s_n x_n + (1-s_n)\bar{x}_{n+1} - p\| \\
&\le s_n\|x_n - p\| + (1-s_n)\|\bar{x}_{n+1} - p\| \\
&\le s_n\|x_n - p\| + (1-s_n)\|\beta_{n,0}x_n + \sum_{i=1}^{\infty}\beta_{n,i}J_{r_n}^{B_i}x_n - p\| \\
&\le s_n\|x_n - p\| + (1-s_n)\Big[\beta_{n,0}\|x_n - p\| + \sum_{i=1}^{\infty}\beta_{n,i}\|J_{r_n}^{B_i}x_n - p\|\Big].
\end{aligned}$$

Hence,

$$\|z_n - p\| \le \|x_n - p\|. \tag{16}$$

We prove that the sequences $\{x_n\}$ is bounded. Using inequalities (11), (16) and the fact that $J_{r_n}^{B_i}$ are nonexpansive, we can write

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n f(x_n) + (1-\alpha_n)z_n - p\| \\
&\le \alpha_n\|f(x_n) - f(p)\| + (1-\alpha_n)\|z_n - p\| + \alpha_n\|f(p) - p\| \\
&\le (1-(1-b)\alpha_n)\|x_n - p\| + \alpha_n\|f(p) - p\| \\
&\le \max\left\{\|x_n - p\|, \frac{\|f(p) - p\|}{1-b}\right\}.
\end{aligned}$$

By induction, it is easy to see that

$$\|x_n - p\| \leq \max\left\{\|x_0 - p\|, \frac{\|f(p) - p\|}{1-b}\right\}, \qquad n \geq 1.$$

Hence, $\{x_n\}$ is bounded, as well as $\{f(x_n)\}$.

Let $k \in \mathbb{N}^*$. From Lemma 4, convexity of $\|.\|^2$, and (11), we have

$$\begin{aligned}
\|z_n - p\|^2 &= \|s_n x_n + (1-s_n)\bar{x}_{n+1} - p\|^2 \\
&\leq s_n\|x_n - p\|^2 + (1-s_n)\|\bar{x}_{n+1} - p\|^2 \\
&\leq s_n\|x_n - p\|^2 + (1-s_n)\|\beta_{n,0}x_n + \sum_{i=1}^{\infty}\beta_{n,i}J_{r_n}^{B_i}x_n - p\|^2 \\
&\leq s_n\|x_n - p\|^2 + (1-s_n)\Big[\beta_{n,0}\|x_n - p\|^2 + \sum_{i=1}^{\infty}\beta_{n,i}\|J_{r_n}^{B_i}x_n - p\|^2 \\
&- \beta_{n,0}\beta_{n,k}g(\|J_{r_n}^{B_k}x_n - x_n\|)\Big] \\
&\leq \|x_n - p\|^2 - (1-s_n)\beta_{n,0}\beta_{n,k}g(\|J_{r_n}^{B_k}x_n - x_n\|).
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n)z_n - p\|^2 \\
&\leq \|\alpha_n(f(x_n) - p) + (1-\alpha_n)(z_n - p)\|^2 \\
&\leq \alpha_n^2\|f(x_n) - p\|^2 + (1-\alpha_n)^2\|z_n - p\|^2 \\
&\quad +2\alpha_n(1-\alpha_n)\|f(x_n) - p\|\|z_n - p\| \\
&\leq \alpha_n^2\|f(x_n) - p\|^2 + (1-\alpha_n)^2\|x_n - p\|^2 \\
&\quad -(1-\alpha_n)^2(1-s_n)\beta_{n,0}\beta_{n,k}g(\|J_{r_n}^{B_k}x_n - x_n\|) \\
&\quad +2\alpha_n(1-\alpha_n)\|f(x_n) - p\|\|x_n - p\|.
\end{aligned}$$

Thus, for every $k \in \mathbb{N}^*$, we get

$$\begin{aligned}
(1-\alpha_n)^2(1-s_n)\beta_{n,0}\beta_{n,k}g(\|J_{r_n}^{B_k}x_n - x_n\|) &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&+\alpha_n^2\|f(x_n) - p\|^2 + 2\alpha_n(1-\alpha_n)\|f(x_n) - p\|\|x_n - p\|. \quad (17)
\end{aligned}$$

Since $\{x_n\}_{n\geq 0}$ and $\{f(x_n)\}_{n\geq 0}$ are bounded, there exists a constant $C > 0$ such that

$$\alpha_n^2\|f(x_n) - p\|^2 + 2\alpha_n(1-\alpha_n)\|f(x_n) - p\|\|x_n - p\| \leq \alpha_n C.$$

Therefor, from (17), we have for every $k \in \mathbb{N}^*$,

$$\begin{aligned}
&(1-\alpha_n)^2(1-s_n)\beta_{n,0}\beta_{n,k}g(\|J_{r_n}^{B_k}x_n - x_n\|) \\
&\qquad\qquad \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n C. \quad (18)
\end{aligned}$$

Observe that $Q_F of$ is a contraction, where Q_F is the sunny nonexpansive retraction from K to F. Indeed, for all $x, y \in K$, we have

$$\|Q_F f(x) - Q_F f(y)\| \leq \|f(x) - f(y)\| \leq b\|x - y\|.$$

Banach's Contraction Mapping Principle guarantees that $Q_F of$ has a unique fixed point, say, $x_1 \in K$. That is, $x_1 = Q_F f(x_1)$. Thus, in view of Lemma 5, it is equivalent to the following variational inequality problem:

$$\langle x_1 - f(x_1), J_\varphi(x_1 - p)\rangle \leq 0, \qquad p \in F.$$

By the uniqueness of the solution of (12), we have, $x_1 = x^*$.

Now let us prove that $\{x_n\}$ converges strongly to x^*. We divide the proof into two cases.

Case 1. Assume that $\{\|x_n - p\|\}$ is a monotonically decreasing sequence. Then $\{\|x_n - p\|\}$ is convergent. Clearly, we have

$$\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \to 0.$$

Then it follows from (18) that

$$\lim_{n\to\infty}(1 - s_n)\beta_{n,0}\beta_{n,k} g(\|J_{r_n}^{B_k} x_n - x_n\|) = 0. \tag{19}$$

Since $\lim_{n\to\infty}\inf(1 - s_n)\beta_{n,0}\beta_{n,k} > 0$ and according to the property of g, we have

$$\lim_{n\to\infty}\|x_n - J_{r_n}^{B_k} x_n\| = 0. \tag{20}$$

By using the resolvent identity (Lemma 7), for any $r > 0$, we conclude that

$$\begin{aligned}
\|x_n - J_r^{B_k} x_n\| &\leq \|x_n - J_{r_n}^{B_k} x_n\| + \|J_{r_n}^{B_k} x_n - J_r^{B_k} x_n\| \\
&\leq \|x_n - J_{r_n}^{B_k} x_n\| \\
&\quad + \|J_r^{B_k}\left(\frac{r}{r_n} x_n + \left(1 - \frac{r}{r_n}\right) J_{r_n}^{B_k} x_n\right) - J_r^{B_k} x_n\| \\
&\leq \|x_n - J_{r_n}^{B_k} x_n\| \\
&\quad + \|\frac{r}{r_n} x_n + \left(1 - \frac{r}{r_n}\right) J_{r_n}^{B_k} x_n - x_n\| \\
&\leq \|x_n - J_{r_n}^{B_k} x_n\| + \left|1 - \frac{r}{r_n}\right| \|J_{r_n}^{B_k} x_n - x_n\|.
\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\|x_n - J_r^{B_k} x_n\| = 0. \tag{21}$$

Next, we prove that $\limsup_{n\to+\infty}\langle x^* - f(x^*), J_\varphi(x^* - x_n)\rangle \leq 0$. Since E is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that x_{n_j} converges weakly to a in K and

$$\limsup_{n\to+\infty}\langle x^* - f(x^*), J_\varphi(x^* - x_n)\rangle = \lim_{j\to+\infty}\langle x^* - f(x^*), J_\varphi(x^* - x_{n_j})\rangle.$$

From (21), the fact that $J_r^{B_k}$, $k \in \mathbb{N}^*$ are nonexpansives and Lemma 1, we obtain $a \in F$. On the other hand, by the assumption that the duality mapping J_φ is weakly continuous and the fact that x^* solves (12), we have

$$\begin{aligned}\limsup_{n\to+\infty}\langle x^* - f(x^*), J_\varphi(x^* - x_n)\rangle &= \lim_{j\to+\infty}\langle x^* - f(x^*), J_\varphi(x^* - x_{n_j})\rangle \\ &= \langle x^* - f(x^*), J_\varphi(x^* - a)\rangle \leq 0.\end{aligned}$$

Finally, we show that $x_n \to x^*$. Since $\Phi(t) = \int_0^t \varphi(\sigma)d\sigma$ for all $t \geq 0$ and φ is a gauge function, $\Phi(kt) \leq k\Phi(t)$ for $0 \geq k \geq 1$. From (11) and Lemma 2, we get that

$$\begin{aligned}\Phi(\|x_{n+1} - x^*\|) &= \Phi(\|\alpha_n\gamma f(x_n) + (1-\alpha_n)z_n - x^*\|) \\ &\leq \Phi(\|\alpha_n(f(x_n) - f(x^*) + (1-\alpha_n)(z_n - x^*)\|) \\ &\quad +\alpha_n\langle x^* - f(x^*), J_\varphi(x^* - x_{n+1})\rangle \\ &\leq \Phi(\alpha_n\|f(x_n) - f(x^*)\| + \|(1-\alpha_n)(z_n - x^*)\|) \\ &\quad +\alpha_n\langle x^* - f(x^*), J_\varphi(x^* - x_{n+1})\rangle \\ &\leq \Phi(\alpha_n\|x_n - x^*\| + (1-\alpha_n)\|z_n - x^*\|) \\ &\quad +\alpha_n\langle x^* - f(x^*), J_\varphi(x^* - x_{n+1})\rangle \\ &\leq \Phi((1-(1-b)\alpha_n)\|x_n - x^*\|) \\ &\quad +\alpha_n\langle x^* - f(x^*), J_\varphi(x^* - x_{n+1})\rangle \\ &\leq (1-(1-b)\alpha_n)\Phi(\|x_n - x^*\|) \\ &\quad +\alpha_n\langle x^* - f(x^*), J_\varphi(x^* - x_{n+1})\rangle.\end{aligned}$$

From Lemma 3, its follows that $x_n \to x^*$.

Case 2. Assume that $\{\|x_n - x^*\|\}$ is not a monotonically decreasing sequence. Set $B_n = \|x_n - x^*\|$ and let τ be a mapping defined by $\tau(n) = \max\{k \in \mathbb{N} : k \leq n,\, B_k \leq B_{k+1}\}$ for all $n \neq n_0$ and some n_0 large enough. Then τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$, and $B_{\tau(n)} \leq B_{\tau(n)+1}$ for $n \geq n_0$. For $i \in \mathbb{N}^*$ from (18), we have

$$(1-\alpha_{\tau(n)})^2(1-s_\tau(n))\beta_{\tau(n),0}\beta_{\tau(n),i}g(\|J^{B_i}_{r_{\tau(n)}}x_{\tau(n)} - x_{\tau(n)}\|) \leq \alpha_{\tau(n)}C \to 0$$

as $n \to \infty$. Furthermore,

$$(1-s_\tau(n))\beta_{\tau(n),0}\beta_{\tau(n),i}g(\|J^{B_i}_{r_{\tau(n)}}x_{\tau(n)} - x_{\tau(n)}\|) \to 0 \text{ as } n \to \infty.$$

Hence,

$$\lim_{n\to\infty}\|J^{B_i}_{r_{\tau(n)}}x_{\tau(n)} - x_{\tau(n)}\| = 0. \tag{22}$$

By the same argument as in case 1, we can show that $x_{\tau(n)}$ is bounded in K and $\limsup\limits_{\tau(n)\to+\infty}\langle x^* - f(x^*), J_\varphi(x^* - x_{\tau(n)})\rangle \leq 0$. For all $n \geq 0$, we have

$$\begin{aligned}0 &\leq \Phi(\|x_{\tau(n)+1} - x^*\|) - \Phi(\|x_{\tau(n)} - x^*\|) \\ &\leq \alpha_{\tau(n)}[-(1-b)\Phi(\|x_{\tau(n)} - x^*\|) + \langle x^* - f(x^*), J_\varphi(x^* - x_{\tau(n)+1})\rangle],\end{aligned}$$

which implies that

$$\Phi(\|x_{\tau(n)} - x^*\|) \leq \frac{1}{1-b}\langle x^* - f(x^*), J_\varphi(x^* - x_{\tau(n)+1})\rangle.$$

Then,

$$\lim_{n\to\infty} \Phi(\|x_{\tau(n)} - x^*\|) = 0.$$

Therefore,

$$\lim_{n\to\infty} B_{\tau(n)} = \lim_{n\to\infty} B_{\tau(n)+1} = 0.$$

Moreover, for all $n \geq n_0$, we have $B_{\tau(n)} \leq B_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is, $n > \tau(n)$) because $B_j > B_{j+1}$ for $\tau(n)+1 \leq j \leq n$. Thus, for all $n \geq n_0$, we have

$$0 \leq B_n \leq \max\{B_{\tau(n)},\ B_{\tau(n)+1}\} = B_{\tau(n)+1}.$$

Hence, $\lim\limits_{n\to\infty} B_n = 0$, that is $\{x_n\}$ converges strongly to x^*. This completes the proof. □

As a consequence of Theorem 2, we have the following theorem.

Theorem 3 *Let E be a uniformly convex real Banach space having a weakly continuous duality map J_φ. Let $f : E \to E$ be an contraction mapping with a constant $b \in [0,1)$. Let B_i, $i \in \mathbb{N}^*$ be an infinite family of multivalued m-accretive operators of E such that $F := \bigcap\limits_{i=1}^{\infty} {B_i}^{-1}(0) \neq \emptyset$. Let $\{\beta_{n,i}\}$, $\{s_n\}$, $\{\alpha_n\}$ be three sequences in $(0,1)$ and $\{r_n\} \subset]0,\infty[$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in K$ by:*

$$\begin{cases} \bar{x}_{n+1} = \beta_{n,0}x_n + \displaystyle\sum_{i=1}^{\infty} \beta_{n,i} J_{r_n}^{B_i} x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\big(s_n x_n + (1-s_n)\bar{x}_{n+1}\big), \end{cases} \tag{23}$$

Suppose the following conditions hold:

$$(i)\ \lim_{n\to\infty} \alpha_n = 0; \qquad (ii)\ \sum_{n=0}^{\infty} \alpha_n = \infty, \quad \sum_{i=0}^{\infty} \beta_{n,i} = 1,$$

$$(iii)\ \liminf_{n\to\infty} r_n > 0 \text{ and } \liminf_{n\to\infty} (1-s_n)\beta_{n,0}\beta_{n,i} > 0 \text{ for all } i \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ generated by (23) converges strongly to $x^* \in F$, which is a unique solution of variational inequality (12) .

Proof. Since B_i, $i \in \mathbb{N}^*$ are m-accretive operators, we conclude that B_i are accretive and satisfy the condition $R(I + rB_i) = E$ for all $r > 0$. Setting $K = E$ in Theorem 2, we obtain the desired result. □

We have the following corollaries.

Corollary 1 *Assume that $E = l_q$, $1 < q < \infty$. Let K be a nonempty, closed and convex subset of E and $f : K \to K$ be an contraction mapping with a constant $b \in [0, 1)$. Let B_i, $i \in \mathbb{N}^*$ be an infinite family of multivalued accretive operators of E such that $F := \bigcap_{i=1}^{\infty} {B_i}^{-1}(0) \neq \emptyset$ and $\bigcap_{i=1}^{\infty} \overline{D(B_i)} \subset K \subset \bigcap_{i=1}^{\infty} R(I + rB_i)$ for all $r > 0$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in K$ by:*

$$\begin{cases} \bar{x}_{n+1} = \beta_{n,0}x_n + \displaystyle\sum_{i=1}^{\infty} \beta_{n,i} J_{r_n}^{B_i} x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)\big(s_n x_n + (1 - s_n)\bar{x}_{n+1}\big), \end{cases} \tag{24}$$

where $\{s_n\}$, $\{\beta_{n,i}\}$, $\{\alpha_n\}$ are sequences in $(0, 1)$ and $\{r_n\} \subset]0, \infty[$ satisfying

$$(i)\ \lim_{n\to\infty} \alpha_n = 0; \qquad (ii)\ \sum_{n=0}^{\infty} \alpha_n = \infty, \quad \sum_{i=0}^{\infty} \beta_{n,i} = 1,$$

$$(iii)\ \liminf_{n\to\infty} r_n > 0 \text{ and } \liminf_{n\to\infty} (1 - s_n)\beta_{n,0}\beta_{n,i} > 0 \text{ for all } i \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ generated by (24) *converges strongly to $x^* \in F$, which is a unique solution of variational inequality* (12).

Proof. Since $E = l_q$, $1 < q < \infty$ are uniformly convex and has a weakly continuous duality map, the proof follows from Theorem 2. □

Corollary 2 *Let H be a real Hilbert space. Let K be a nonempty, closed and convex subset of E and $f : K \to K$ be an contraction mapping with a constant $b \in [0, 1)$. Let B_i, $i \in \mathbb{N}^*$ be an infinite family of multivalued monotone operators of E such that $F := \bigcap_{i=1}^{\infty} {B_i}^{-1}(0) \neq \emptyset$ and $\bigcap_{i=1}^{\infty} \overline{D(B_i)} \subset K \subset \bigcap_{i=1}^{\infty} R(I + rB_i)$ for all $r > 0$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in K$ by*

$$\begin{cases} \bar{x}_{n+1} = \beta_{n,0}x_n + \displaystyle\sum_{i=1}^{\infty} \beta_{n,i} J_{r_n}^{B_i} x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)\big(s_n x_n + (1 - s_n)\bar{x}_{n+1}\big), \end{cases} \tag{25}$$

where $\{s_n\}$, $\{\beta_{n,i}\}$, $\{\alpha_n\}$ are sequences in $(0, 1)$ and $\{r_n\} \subset]0, \infty[$ satisfying

$$(i)\ \lim_{n\to\infty} \alpha_n = 0; \qquad (ii)\ \sum_{n=0}^{\infty} \alpha_n = \infty, \quad \sum_{i=0}^{\infty} \beta_{n,i} = 1,$$

$$(iii)\ \liminf_{n\to\infty} r_n > 0 \text{ and } \liminf_{n\to\infty} (1 - s_n)\beta_{n,0}\beta_{n,i} > 0 \text{ for all } i \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ generated by (25) converges strongly to $x^* \in F$, which is a unique solution of variational inequality

$$\langle x^* - f(x^*), x^* - p\rangle \leq 0, \qquad p \in F. \tag{26}$$

3 Application to equilibrium problems

In this section, we apply Theorem 2 to equilibrium problems in Hilbert spaces.

Let H be a real Hilbert space and let C be a nonempty, closed, and convex subset of H. Let F be a function from $C \times C$ into $\mathbb{R}$. The equilibrium problem for F is to find $x \in C$ such that

$$F(x, y) \geq 0 \qquad \text{for all } y \in C. \tag{27}$$

The set of solutions is denoted by $EP(F)$.

Equilibrium problems which were introduced by Blum and Oettli [5] have a great impact and influence on the development of several branches of pure and applied sciences.

To solve the equilibrium problem for a function $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:
$(A1)$ $F(x, x) = 0$ for all $x \in C$;
$(A2)$ F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
$(A3)$ for each $x, y, z \in C$, $\lim_{t\to 0} F(tz + (1-t)x, y) \leq F(x, y)$;
$(A4)$ for each $x \in C$, the function $y \to F(x, y)$ is convex and lower semicontinuous.

Lemma 9 ([7]) *Assume that $F : C \times C \to \mathbb{R}$ satisfies $(A1) - (A4)$. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C :\ F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, y \in C \right\}, \qquad x \in H.$$

Then
1. T_r is single-valued;
2. T_r is firmly nonexpansive, i.e., $\|T_r(x) - T_r(y)\|^2 \leq \langle T_r x - T_r y, x - y\rangle$ for any $x, y \in H$;
3. $F(T_r) = EP(F)$;
4. $EP(F)$ is closed and convex.

The following lemma appears implicitly in [25].

Lemma 10 ([25]) *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $F: C \times C \to \mathbb{R}$ satisfies $(A1)-(A4)$. Let A_F be a set-valued mapping from H into itself defined by*

$$A_F x = \begin{cases} \{z \in H,\ F(x,y) \geq \langle y-x, z\rangle,\ y \in C\},\ x \in C, \\ \emptyset,\ x \notin C. \end{cases} \tag{28}$$

Then $EP(F) = {A_F}^{-1}0$, and A_F is a maximal monotone operator with $D(A_F) \subset C$. Furthermore, for any $x \in H$ and $r > 0$, the map T_r defined in Lemma 9 coincides with the resolvent of A_F, i.e, $T_r x = \big(I + rA_F\big)^{-1}x$.

Using Theorem 2, we prove a strong convergence theorem for an equilibrium problem in a Hilbert space.

Theorem 4 *Let H be a real Hilbert space and let function $F: H \times H \to \mathbb{R}$ satisfying $(A1)-(A4)$ be such that $EP(F) \neq \emptyset$. Let $f: H \to H$ be a contraction mapping with a constant $b \in [0,1)$. Let $\{x_n\}$ and $\{u_n\}$ be sequences defined iteratively from arbitrary $x_0 \in H$ by*

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \textit{for all } y \in H, \\ \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)u_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\big(s_n x_n + (1-s_n)\bar{x}_{n+1}\big), \end{cases} \tag{29}$$

where $\{s_n\}$, $\{\beta_{n,i}\}$, $\{\alpha_n\}$ are sequences in $(0,1)$ and $\{r_n\} \subset]0,\infty[$ satisfying

$$(i)\ \lim_{n\to\infty} \alpha_n = 0; \qquad (ii)\ \sum_{n=0}^{\infty} \alpha_n = \infty, \quad \sum_{i=0}^{\infty} \beta_{n,i} = 1,$$

$$(iii)\ \liminf_{n\to\infty} r_n > 0 \ \textit{and}\ \liminf_{n\to\infty} (1-s_n)\beta_{n,0}\beta_{n,i} > 0 \ \textit{for all } i \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ generated by (29) converges strongly to $x^* \in EP(F)$, which is a unique solution of variational inequality

$$\langle x^* - f(x^*), x^* - p\rangle \leq 0, \qquad p \in EP(F). \tag{30}$$

Proof. Since $F: H \times H \to \mathbb{R}$ satisfies $(A1)-(A4)$, the mapping A_F defined by Lemma 10 is a maximal and monotone operator. Put $B = A_F$ in Theorem 3 (with i=1). Then, $u_n = T_{r_n}x_n = J^B_{r_n}x_n$. Therefore, we obtain the desired results. □

4 Application to convex minimization problems

In this section, we apply our results to convex minimization involving an infinite family of semicontinuous and convex function in a Hilbert space.

Problem 1 *Let H be a real Hilbert space and $g_i : H \to \mathbb{R} \cup \{\infty\}$, $i \in \mathbb{N}^*$ be proper lower semi-continuous and convex functions. We consider the following convex minimization problem: find $x^* \in H$ such that*

$$x^* \in \bigcap_{i=1}^{\infty} argmin_{x\in H}\, g_i(x), \tag{31}$$

where $argmin_{x\in H}\, g_i(x)$ denotes the set of minimizers of g_i.

Using Theorem 2, we obtain the following result.

Theorem 5 *Let H be a real Hilbert space and $f : H \to H$ be an contraction mapping with a constant $b \in [0, 1)$. Let $g_i : H \to \mathbb{R} \cup \{\infty\}$, $i \in \mathbb{N}^*$ be proper lower semi continuous and convex functions such that $F := \bigcap\limits_{i=1}^{\infty} \partial {g_i}^{-1}(0) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in H$ by*

$$\begin{cases} \bar{x}_{n+1} = \beta_{n,0}x_n + \displaystyle\sum_{i=1}^{\infty} \beta_{n,i} J_{r_n}^{\partial g_i} x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\big(s_n x_n + (1-s_n)\bar{x}_{n+1}\big), \end{cases} \tag{32}$$

where $\{s_n\}$, $\{\beta_{n,i}\}$, $\{\alpha_n\}$ are sequences in $(0, 1)$ and $\{r_n\} \subset]0, \infty[$ satisfying

$$(i)\ \lim_{n\to\infty} \alpha_n = 0; \qquad (ii)\ \sum_{n=0}^{\infty} \alpha_n = \infty, \quad \sum_{i=0}^{\infty} \beta_{n,i} = 1,$$

$$(iii)\ \liminf_{n\to\infty} r_n > 0 \text{ and } \liminf_{n\to\infty} (1-s_n)\beta_{n,0}\beta_{n,i} > 0 \text{ for all } i \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ generated by (32) converges strongly to a solution of Problem 1 which is a unique solution of variational inequality

$$\langle x^* - f(x^*), x^* - p\rangle \leq 0, \qquad p \in F. \tag{33}$$

Proof. For each $i \in \mathbb{N}^*$, set $B_i = \partial g_i$ in Theorem 2. Then $\partial {g_i}^{-1}(0) = {B_i}^{-1}(0)$ for all $i \in \mathbb{N}^*$, and hence $\bigcap\limits_{i=1}^{\infty} \partial {g_i}^{-1}(0) = \bigcap\limits_{i=1}^{\infty} {B_i}^{-1}(0)$. Furthermore, each B_i is maximal monotone (see, e.g., Minty [19]). Therefore, the proof follows from Theorem 2. □

Remark 2 *Our results are applicable for the family of pseudo-contactive mappings. Moreover, the theorems in this paper complement the explicit midpoint rule and proximal point algorithm by proposing strong convergence to zero of monotone mapping, and extend and unify some results (see, e.g., Rockafellar [25], Auzinger [1], Bader[2]).*

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