Armenian Journal of Mathematics
Volume 12, Number 6, 2020, 1–17

Generic lightlike submanifolds of an indefinite Kaehler manifold with an (ℓ, m)-type metric connection

C. W. Lee and J. W. Lee

Abstract. We study generic lightlike submanifolds M of an indefinite Kaehler manifold $\bar{M}$ or an indefinite complex space form $\bar{M}(c)$ with an (ℓ, m)-type metric connection subject such that the characteristic vector field ζ of $\bar{M}$ belongs to our screen distribution $S(TM)$ of M.

Key Words: Generic lightlike submanifold, (ℓ, m)-type metric connection, Indefinite Kaehler manifold, Indefinite complex space form
Mathematics Subject Classification 2010: 53C25, 53C40, 53C50

1 Introduction

Let (M, g) be an m-dimensional lightlike submanifold of an indefinite Kaehler manifold $(\bar{M}\bar{g})$ of dimension $(m+n)$. Then the radical distribution $Rad(TM)$ $= TM \cap TM^{\perp}$ of M is a vector subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$ of rank r $(1 \leq r \leq \min\{m, n\})$. Due to [2], in general, we can take two complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and in $TM^{\perp}$, respectively, which are called the *screen* and *co-screen* distributions of M, such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \quad TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $S(TM)^{*} = TM/Rad(TM)$ due to Kupeli [13]. Thus, all screen distributions $S(TM)$ are mutually isomorphic. Therefore, the following definition is well-defined:

A lightlike submanifold M of an indefinite Kaehler manifold $\bar{M}$ with an indefinite almost complex structure J is called a *generic submanifold* [10] if there exists a screen distribution $S(TM)$ such that

$$J(S(TM)^{\perp}) \subset S(TM), \tag{1.1}$$

where the symbol $S(TM)^{\perp}$ denotes the orthogonal complement of $S(TM)$ in the tangent bundle $T\bar{M}$ of $\bar{M}$ such that $T\bar{M} = S(TM) \oplus_{orth} S(TM)^{\perp}$. The notion of generic lightlike submanifolds was studied by several authors (see, for example, [3, 5, 6, 11]). Lightlike hypersurfaces of an indefinite almost complex manifold are important examples of the generic lightlike submanifold.

The notion of symmetric connection of type (ℓ, m) on semi-Riemannian manifolds was introduced by the author of [7, 8] as follows:

From now and in the sequel, we denote by $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ the vector fields on $\bar{M}$. A linear connection $\bar{\nabla}$ on a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be a *symmetric connection of type* $(\ell,\, m)$ if its torsion tensor $\bar{T}$ satisfies

$$\bar{T}(\bar{X}, \bar{Y}) = \ell\{\theta(\bar{Y})\bar{X} - \theta(\bar{X})\bar{Y}\} + m\{\theta(\bar{Y})J\bar{X} - \theta(\bar{X})J\bar{Y}\}, \tag{1.2}$$

where ℓ and m are smooth functions, J is a tensor field of type $(1, 1)$, and θ is a 1-form associated with a smooth vector field ζ, called a *characteristic vector field*, by $\theta(\bar{X}) = \bar{g}(\bar{X}, \zeta)$. Moreover, if this connection is a metric connection, *i.e.*, satisfies $\bar{\nabla}\bar{g} = 0$, then $\bar{\nabla}$ is called a *symmetric metric connection of type* $(\ell,\, m)$ or an $(\ell,\, m)$*-type metric connection*.

In case $(\ell, m) = (1,\, 0)$, this connection becomes a semi-symmetric metric connection, introduced by Hayden [4] and Yano [14]. If $(\ell, m) = (0,\, 1)$, this connection becomes a quarter-symmetric metric connection, introduced by Yano-Imai [15]. In this paper, we shall assume that $(\ell, m) \neq (0, 0)$ and, without loss of generality, that the vector field ζ is unit spacelike.

Remark 1 *Denote by* $\widetilde{\nabla}$ *the Levi-Civita connection of a semi-Riemannian manifold* $(\bar{M}, \bar{g})$ *with respect to* $\bar{g}$. *It is known [9] that a linear connection* $\bar{\nabla}$ *on* $\bar{M}$ *is an* (ℓ, m)*-type metric connection if and only if it satisfies*

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \widetilde{\nabla}_{\bar{X}}\bar{Y} + \ell\{\theta(\bar{Y})\bar{X} - \bar{g}(\bar{X}, \bar{Y})\zeta\} - m\theta(\bar{X})J\bar{Y}. \tag{1.3}$$

The object of this paper is to study generic lightlike submanifolds M of an indefinite Kaehler manifold $\bar{M}$ with an (ℓ, m)-type metric connection $\bar{\nabla}$ subject to the condition that the characteristic vector field ζ of $\bar{M}$ belongs to our screen distribution $S(TM)$ of M. In Section 3, we provide several new results on such a generic lightlike submanifold. In Section 4, we characterize generic lightlike submanifolds of an indefinite complex space form $\bar{M}(c)$ with an (ℓ, m)-type metric connection subject such that ζ belongs to $S(TM)$.

2 (ℓ, m)-type metric connections

Let $\bar{M} = (\bar{M},\, \bar{g},\, J)$ be an indedinite Kaehler manifold where $\bar{g}$ is a semi-Riemannian metric and J is an indefinite almost complex structure;

$$J^2\bar{X} = -\bar{X}, \qquad \bar{g}(J\bar{X}, J\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}), \qquad (\widetilde{\nabla}_{\bar{X}}J)\bar{Y} = 0. \tag{2.1}$$

Replacing the Levi-Civita connection $\widetilde{\nabla}$ by the (ℓ, m)-type metric connection $\bar{\nabla}$, the third equation in (2.1) is reduced to

$$(\bar{\nabla}_{\bar{X}}J)(\bar{Y}) = \ell\{\theta(J\bar{Y})\bar{X} - \theta(\bar{Y})J\bar{X} - \bar{g}(\bar{X}, J\bar{Y})\zeta + g(\bar{X}, \bar{Y})J\zeta\}. \quad (2.2)$$

Let (M, g) be an m-dimensional lightlike submanifold of an indefinite Kaehler manifold $(\bar{M}, \bar{g})$, of dimension $(m+n)$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. Also denote by $(2.1)_i$ the i-th equation of (2.1). We use the same notations for any others. Let X, Y and Z be the vector fields on M, unless otherwise specified. We use the following range of indices:

$$i,\ j,\ k,\ ... \in \{1,\ ...,\ r\}, \qquad a,\ b,\ c,\ ... \in \{r\,+\,1,\ ...,\ n\}.$$

Let $tr(TM)$ and $ltr(TM)$ be complementary vector bundles to TM in $T\bar{M}_{|M}$ and $TM^{\perp}$ in $S(TM)^{\perp}$, respectively, and let $\{N_1, \cdots, N_r\}$ be a null basis of $ltr(TM)_{|\mathcal{U}}$ where $\mathcal{U}$ is a coordinate neighborhood of M such that

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0,$$

and $\{\xi_1, \cdots, \xi_r\}$ is a null basis of $Rad(TM)_{|\mathcal{U}}$. Then we have

$$\begin{aligned} T\bar{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}). \end{aligned}$$

A lightlike submanifold $M = (M, g, S(TM), S(TM^{\perp}))$ of $\bar{M}$ is called an *r-lightlike submanifold* [2] if $1 \le r < \min\{m, n\}$. For an r-lightlike M, we see that $S(TM) \ne \{0\}$ and $S(TM^{\perp}) \ne \{0\}$. In the sequel, by saying that M is a lightlike submanifold we shall mean that it is an r-lightlike submanifold with following local quasi-orthonormal field of frames of $\bar{M}$:

$$\{\xi_1, \cdots, \xi_r\,,\ N_1, \cdots, N_r\,,\ F_{r+1}, \cdots, F_m\,,\ E_{r+1}, \cdots, E_n\},$$

where $\{F_{r+1}, \cdots, F_m\}$ and $\{E_{r+1}, \cdots, E_n\}$ are orthonormal basis of $S(TM)$ and $S(TM^{\perp})$, respectively. Denote $\epsilon_a = \bar{g}(E_a, E_a)$. Then $\epsilon_a\delta_{ab} = \bar{g}(E_a, E_b)$.

Let P be the projection morphism of TM on $S(TM)$. The local Gauss-Weingarten formulae of M and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{r} h_i^{\ell}(X, Y)N_i + \sum_{a=r+1}^{n} h_a^{s}(X, Y)E_a, \quad (2.3)$$

$$\bar{\nabla}_X N_i = -A_{N_i}X + \sum_{j=1}^{r} \tau_{ij}(X)N_j + \sum_{a=r+1}^{n} \rho_{ia}(X)E_a, \quad (2.4)$$

$$\bar{\nabla}_X E_a = -A_{E_a}X + \sum_{i=1}^{r} \lambda_{ai}(X)N_i + \sum_{b=r+1}^{n} \mu_{ab}(X)E_b, \quad (2.5)$$

$$\nabla_X PY = \nabla^*_X PY + \sum_{i=1}^{r} h^*_i(X, PY)\xi_i, \tag{2.6}$$

$$\nabla_X \xi_i = -A^*_{\xi_i} X - \sum_{j=1}^{r} \tau_{ji}(X)\xi_j, \tag{2.7}$$

where ∇ and ∇^* are induced linear connections on M and $S(TM)$, respectively, h^ℓ_i and h^s_a are called the *local second fundamental forms* on M, h^*_i's are called the *local second fundamental forms* on $S(TM)$. A_{N_i}, A_{E_a} and $A^*_{\xi_i}$ are called the *shape operators*, and τ_{ij}, ρ_{ia}, λ_{ai} and μ_{ab} are 1-forms on M.

Let M be a generic lightlike submanifold of $\bar{M}$. From (1.1), we see that the distributions $J(Rad(TM))$, $J(ltr(TM))$ and $J(S(TM^\perp))$ are subbundles of $S(TM)$. Thus, there exist two non-degenerate almost complex distributions H_o and H with respect to J, *i.e.*, $J(H_o) = H_o$ and $J(H) = H$, such that

$$S(TM) = \{J(Rad(TM)) \oplus J(ltr(TM))\} \oplus_{orth} J(S(TM^\perp)) \oplus_{orth} H_o,$$
$$H = Rad(TM) \oplus_{orth} J(Rad(TM)) \oplus_{orth} H_o.$$

In this case, the tangent bundle TM of M is decomposed as follows:

$$TM = H \oplus J(ltr(TM)) \oplus_{orth} J(S(TM^\perp)). \tag{2.8}$$

Consider r-th local null vector fields U_i and V_i, $(n-r)$-th local non-null unit vector fields W_a, and their 1-forms u_i, v_i and w_a defined by

$$U_i = -JN_i, \qquad V_i = -J\xi_i, \qquad W_a = -JE_a, \tag{2.9}$$

$$u_i(X) = g(X, V_i), \quad v_i(X) = g(X, U_i), \quad w_a(X) = \epsilon_a g(X, W_a). \tag{2.10}$$

Denote by S the projection morphism of TM on H and by F the tensor field of type $(1,1)$ globally defined on M by $F = J \circ S$. Then JX is expressed as

$$JX = FX + \sum_{i=1}^{r} u_i(X)N_i + \sum_{a=r+1}^{n} w_a(X)E_a. \tag{2.11}$$

Applying J to (2.11) and using $(2.1)_1$, (2.9) and (2.11) we obtain

$$F^2X = -X + \sum_{i=1}^{r} u_i(X)U_i + \sum_{a=r+1}^{n} w_a(X)W_a. \tag{2.12}$$

By $(2.1)_2$ and (2.11) we have

$$g(FX, FY) = g(X, Y) - \sum_{i=1}^{r}\{u_i(X)v_i(Y) + u_i(Y)v_i(X)\} - \sum_{a=r+1}^{n} \epsilon_a w_a(X)w_a(Y). \tag{2.13}$$

According to (1.2), (1.3), (2.3), and (2.11) we see that

$$(\nabla_X g)(Y,Z) = \sum_{i=1}^{r}\{h_i^\ell(X,Y)\eta_i(Z) + h_i^\ell(X,Z)\eta_i(Y)\}, \tag{2.14}$$
$$T(X,Y) = \ell\{\theta(Y)X - \theta(X)Y\} + m\{\theta(Y)FX - \theta(X)FY\}, \tag{2.15}$$
$$h_i^\ell(X,Y) - h_i^\ell(Y,X) = m\{\theta(Y)u_i(X) - \theta(X)u_i(Y)\}, \tag{2.16}$$
$$h_a^s(X,Y) - h_a^s(Y,X) = m\{\theta(Y)w_a(X) - \theta(X)w_a(Y)\}. \tag{2.17}$$

where η_i's are 1-forms such that $\eta_i(X) = \bar{g}(X, N_i)$. From the facts that $h_i^\ell(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi_i)$ and $\epsilon_a h_a^s(X,Y) = \bar{g}(\bar{\nabla}_X Y, E_a)$, we know that h_i^ℓ and h_a^s are independent of the choice of $S(TM)$. The above local second fundamental forms are related to their shape operators by

$$h_i^\ell(X,Y) = g(A^*_{\xi_i}X, Y) - \sum_{k=1}^{r} h_k^\ell(X,\xi_i)\eta_k(Y), \tag{2.18}$$
$$\epsilon_a h_a^s(X,Y) = g(A_{E_a}X, Y) - \sum_{k=1}^{r} \lambda_{ak}(X)\eta_k(Y), \tag{2.19}$$
$$h_i^*(X, PY) = g(A_{N_i}X, PY). \tag{2.20}$$

Applying $\bar{\nabla}_X$ to $\bar{g}(E_a, E_b) = \epsilon\delta_{ab}$, $g(\xi_i, \xi_j) = 0$, $\bar{g}(\xi_i, E_a) = 0$, $\bar{g}(N_i, N_j) = 0$ and $\bar{g}(N_i, E_a) = 0$ by turns, we obtain $\epsilon_b\mu_{ab} + \epsilon_a\mu_{ba} = 0$ and

$$\begin{aligned} &h_i^\ell(X,\xi_j) + h_j^\ell(X,\xi_i) = 0, \qquad && h_a^s(X,\xi_i) = -\epsilon_a\lambda_{ai}(X), \\ &\eta_j(A_{N_i}X) + \eta_i(A_{N_j}X) = 0, && \bar{g}(A_{E_a}X, N_i) = \epsilon_a\rho_{ia}(X). \end{aligned} \tag{2.21}$$

Furthermore, using $(2.21)_1$ we see that

$$h_i^\ell(X,\xi_i) = 0, \qquad h_i^\ell(\xi_j,\xi_k) = 0, \qquad A^*_{\xi_i}\xi_i = 0. \tag{2.22}$$

Definition 1 *We say that a lightlike submanifold M is*

(1) *irrotational [13] if $\bar{\nabla}_X\xi_i \in \Gamma(TM)$ for all $i \in \{1, \cdots, r\}$;*

(2) *solenoidal [12] if A_{E_a} and A_{N_i} are $S(TM)$-valued;*

(3) *statical [12] if M is both irrotational and solenoidal.*

Remark 2 *From (2.3) and $(2.21)_2$, the item (1) is equivalent to*

$$h_j^\ell(X,\xi_i) = 0, \qquad h_a^s(X,\xi_i) = \lambda_{ai}(X) = 0. \tag{2.23}$$

By $(2.21)_4$ the item (2) is equivalent to

$$\eta_j(A_{N_i}X) = 0, \qquad \rho_{ia}(X) = \eta_i(A_{E_a}X) = 0. \tag{2.24}$$

Now we shall assume that the characteristic vector field ζ belongs to the screen distribution $S(TM)$. Applying $\bar{\nabla}_X$ to $(2.9)_{1,2,3}$ and (2.11) by turns and using (2.2), (2.3) ∼ (2.7), (2.18) ∼ (2.20) and (2.9) ∼ (2.11) we get

$$\begin{aligned}
&h_j^\ell(X,U_i)=h_i^*(X,V_j)-\ell\theta(V_j)\eta_i(X),\\
&\epsilon_a h_a^s(X,U_i)=h_i^*(X,W_a)-\ell\theta(W_a)\eta_i(X),\\
&h_j^\ell(X,V_i)=h_i^\ell(X,V_j),\\
&h_a^s(X,V_i)=\epsilon_a h_i^\ell(X,W_a),\\
&\epsilon_b h_b^s(X,W_a)=\epsilon_a h_a^s(X,W_b),
\end{aligned}\tag{2.25}$$

$$\begin{aligned}
\nabla_X U_i \;=\;& F(A_{N_i}X)+\sum_{j=1}^r \tau_{ij}(X)U_j+\sum_{a=r+1}^n \rho_{ia}(X)W_a \qquad (2.26)\\
&+\ell\{\theta(U_i)X-v_i(X)\zeta-\eta_i(X)F\zeta\},\\
\nabla_X V_i \;=\;& F(A^*_{\xi_i}X)-\sum_{j=1}^r \tau_{ji}(X)V_j+\sum_{j=1}^r h_j^\ell(X,\xi_i)U_j \qquad (2.27)\\
&-\sum_{a=r+1}^n \epsilon_a\lambda_{ai}(X)W_a+\ell\{\theta(V_i)X-u_i(X)\zeta\},\\
\nabla_X W_a \;=\;& F(A_{E_a}X)+\sum_{i=1}^r \lambda_{ai}(X)U_i+\sum_{b=r+1}^n \mu_{ab}(X)W_b, \qquad (2.28)\\
&+\ell\{\theta(W_a)X-\epsilon_a w_a(X)\zeta\},\\
(\nabla_X F)Y \;=\;& \sum_{i=1}^r u_i(Y)A_{N_i}X+\sum_{a=r+1}^n w_a(Y)A_{E_a}X \qquad (2.29)\\
&-\sum_{i=1}^r h_i^\ell(X,Y)U_i-\sum_{a=r+1}^n h_a^s(X,Y)W_a\\
&+\ell\{\theta(FY)X-\theta(Y)FX\\
&\qquad -\bar{g}(X,JY)\zeta+g(X,Y)F\zeta\}.
\end{aligned}$$

3 Some results

Theorem 1 *Let M be a generic lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ with an (ℓ, m)-type metric connection $\bar{\nabla}$ such that ζ belongs to $S(TM)$. If F is parallel with respect to the connection ∇, then*

(1) $\ell = 0$ and $\bar{\nabla}$ is a quarter-symmetric metric connection,

(2) M is statical,

(3) H, $J(tr(TM))$ and $J(S(TM^\perp))$ are parallel distributions on M,

(4) M is locally a product manifold $M_r \times M_{n-r} \times M^\sharp$, where M_r, M_{n-r} and $M^\sharp$ are leaves of $J(tr(TM))$, $J(S(TM^\perp))$ and H, respectively.

Proof. (1) Replacing Y by ξ_j in (2.29) in order that $\nabla_X F = 0$, we get

$$\sum_{k=1}^{r} h_k^\ell(X,\xi_j)U_k + \sum_{b=r+1}^{n} h_b^s(X,\xi_j)W_b + \ell\{\theta(V_j)X - u_j(X)\zeta\} = 0. \tag{3.1}$$

Taking the scalar product of U_i and (3.1) and then taking in turns $X = V_j$ and $X = U_j$ in the resulting equation, we obtain

$$\ell\theta(V_i) = 0, \qquad \ell\theta(U_i) = 0.$$

Taking the scalar product of V_i and W_a in (3.1) in turns, it becomes

$$h_i^\ell(X,\xi_j) = 0, \qquad \epsilon_a h_a^s(X,\xi_j) = \ell\theta(W_a)u_j(X). \tag{3.2}$$

Replacing Y by W_a in (2.29) and using the fact that $FW_a = 0$, we have

$$A_{E_a}X = \sum_{i=1}^{r} h_i^\ell(X,W_a)U_i + \sum_{b=r+1}^{n} h_b^s(X,W_a)W_b + \ell\{\theta(W_a)FX - \epsilon_a w_a(X)F\zeta\}.$$

Taking the scalar product of U_i and the above equation and using (2.19), we obtain

$$\epsilon_a h_a^s(X,U_i) = -\ell\theta(W_a)\eta_i(X).$$

After substitution $X = \xi_j$ into this equation, it becomes $\epsilon_a h_a^s(\xi_j,U_i) = -\ell\theta(W_a)\delta_{ij}$. Further, substituting $X = U_i$ into $(3.2)_2$, we get $\epsilon_a h_a^s(U_i,\xi_j) = \ell\theta(W_a)\delta_{ij}$. From (2.17), we see that $h_a^s(U_i,\xi_j) = h_a^s(\xi_j,U_i)$. Thus, $\ell\theta(W_a) = 0$, and we have (2.23). Hence, M is irrotational. Eq. (3.1) reduces to $\ell u_j(X) = 0$. It follows that $\ell = 0$.

(2) Taking the scalar product of N_j and (2.29) and using the fact that $\ell = 0$, we get

$$\sum_{k=1}^{r} u_k(Y)\eta_j(A_{N_k}X) + \sum_{b=r+1}^{n} w_b(Y)\eta_j(A_{E_b}X) = 0.$$

Substituting $Y = U_i$ and $Y = W_a$ into this equation, we obtain (2.24). Thus, M is solenoidal, and, therefore, M is statical.

(3) Taking the scalar product of V_i and (2.29), as well as the scalar product of W_b and (2.29), we get

$$h_i^\ell(X,Y) = \sum_{j=1}^{r} u_j(Y)u_i(A_{N_j}X) + \sum_{a=r+1}^{n} w_a(Y)u_i(A_{E_a}X),$$

$$\epsilon_a h_a^s(X,Y) = \sum_{j=1}^{r} u_i(Y)w_a(A_{N_i}X) + \sum_{b=r+1}^{n} w_b(Y)w_a(A_{E_b}X).$$

Putting $Y = V_j$ and $Y = FZ$ in turns into these two equations, we obtain

$$h_i^\ell(X, V_j) = 0, \qquad h_i^\ell(X, FZ) = 0,$$
$$h_a^s(X, V_j) = 0, \qquad h_a^s(X, FZ) = 0.$$

Using (2.7), (2.11), (2.18), (2.19), (2.23), $(2.25)_4$, (2.27), and (2.28), we derive

$$g(\nabla_X \xi_i, V_j) = -h_i^\ell(X, V_j) = 0, \quad g(\nabla_X \xi_i, W_a) = -\epsilon_a h_a^s(X, V_i) = 0,$$
$$g(\nabla_X V_i, V_j) = h_j^\ell(X, \xi_i) = 0, \qquad g(\nabla_X V_i, W_a) = h_a^s(X, \xi_i) = 0,$$
$$g(\nabla_X Z_o, V_j) = h_j^\ell(X, FZ_o) = 0, \quad g(\nabla_X Z_o, W_a) = h_a^s(X, FZ_o) = 0,$$

for all $Z_o \in \Gamma(H_o)$. It follows that H is a parallel distribution on M, *i.e.*,

$$\nabla_X Y \in \Gamma(H), \quad \forall\, X \in \Gamma(TM), \quad \forall\, Y \in \Gamma(H).$$

Further, substituting $Y = U_i$ and $Y = W_a$ into (2.29) in turns, we have

$$A_{N_i} X = \sum_{j=1}^{r} h_j^\ell(X, U_i) U_j + \sum_{a=r+1}^{n} h_a^s(X, U_i) W_a, \tag{3.3}$$
$$A_{E_a} X = \sum_{i=1}^{r} h_i^\ell(X, W_a) U_j + \sum_{b=r+1}^{n} h_b^s(X, W_a) W_b.$$

Applying F to the last two equations, we obtain

$$F(A_{N_i} X) = 0, \qquad F(A_{E_a} X) = 0,$$

respectively. From the last two equations, (2.26) and (2.28), it follows that

$$\nabla_X U_i = \sum_{j=1}^{r} \tau_{ij}(X) U_j, \qquad \nabla_X W_a = \sum_{b=r+1}^{n} \mu_{ab}(X) W_b. \tag{3.4}$$

Thus, $J(tr(TM))$ and $J(S(TM^\perp))$ are parallel distributions on M, *i.e.*,

$$\nabla_X U_i \in \Gamma(J(tr(TM))), \quad \nabla_X W_a \in \Gamma(J(S(TM^\perp))), \quad \forall\, X \in \Gamma(TM).$$

(4) As $J(tr(TM))$, $J(S(TM^\perp))$ and H are parallel distributions satisfying (2.8), by the decomposition theorem [1] M is locally a product manifold $M_r \times M_{n-r} \times M^\sharp$, where M_r, M_{n-r} and $M^\sharp$ are leaves of the distributions $J(tr(TM))$, $J(S(TM^\perp))$ and H, respectively. □

Theorem 2 *Let M be a generic lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ with an (ℓ, m)-type metric connection subject such that ζ belongs to $S(TM)$. If U_i's are parallel with respect to the connection ∇ and the 1-forms ρ_{ia} satisfying $\rho_{ia} = 0$, then M is solenoidal and*

$$(X\ell)\theta(U_i) + \ell(\bar{\nabla}_X \theta)(U_i) = 0. \tag{3.5}$$

Proof. Taking the scalar product of W_a and (2.26) with $\nabla_X U_i = 0$ and using the fact that $\rho_{ia} = 0$, we get $\ell\{\epsilon_a \theta(U_i) w_a(X) - \theta(W_a) v_i(X)\} = 0$. Taking $X = W_a$ and $X = V_i$ in this equation in turns, we have

$$\ell\theta(U_i) = 0, \qquad \ell\theta(W_a) = 0. \tag{3.6}$$

Taking the scalar product of U_j in (2.26), we obtain $\eta_j(A_{N_i} X) = 0$. From this and the fact that $\rho_{ia}(X) = \eta_i(A_{E_a} X) = 0$, we see that M is solenoidal. Applying $\bar{\nabla}_X$ to $\ell\theta(U_i) = 0$ and using the fact that $\nabla_X U_i = 0$, we get (3.5). □

Theorem 3 *Let M be a generic lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ with an (ℓ, m)-type metric connection subject such that ζ belongs to $S(TM)$. If V_i's are parallel with respect to ∇ and the 1-form λ_{ai} satisfy $\lambda_{ai} = 0$, then* (1) *M is irrotational,* (2) *$\ell = 0$ and* (3) *$\tau_{ij} = 0$.*

Proof. Taking the scalar product of W_a in (2.27) with $\nabla_X V_i = 0$ and using the fact that $\lambda_{ai} = 0$, we get $\ell\{\epsilon_a \theta(V_i) w_a(X) - \theta(W_a) u_i(X)\} = 0$. Substituting $X = W_a$ and $X = U_i$ into this equation in turns, we get

$$\ell\theta(V_i) = 0, \qquad \ell\theta(W_a) = 0. \tag{3.7}$$

Taking the scalar product of V_j and (2.27), we obtain $h^\ell_j(X, \xi_i) = 0$. From this and the fact that $\lambda_{ai}(X) = h^s_a(X, \xi_i) = 0$, we see that M is irrotational. Taking in turns the scalar product of N_j, U_j, ζ and (2.27) with $\nabla_X V_i = 0$ and using (2.23) and $(3.7)_1$, it becomes

$$h^\ell_i(X, U_j) = 0, \qquad \tau_{ij}(X) = -\ell\theta(U_i) u_j(X), \tag{3.8}$$

$$g(F(A^*_{\xi_i} X), \zeta) = \ell u_i(X). \tag{3.9}$$

Replacing Y by U_j in (2.16) and using $(3.8)_1$, we have

$$h^\ell_i(U_j, X) = m\{\theta(X)\delta_{ij} - \theta(U_j) u_i(X)\}. \tag{3.10}$$

From this, (2.18), (2.23), and the fact that $S(TM)$ is non-degenerate, we get

$$A^*_{\xi_i} U_j = m\{\delta_{ij}\zeta - \theta(U_j) V_i\}.$$

Taking $X = U_i$ in (3.9) and using the last equation, we obtain

$$\ell = g(F(A^*_{\xi_i} U_i), \zeta) = m\{g(F\zeta, \zeta) - \theta(U_i) g(\xi_i, \zeta)\} = 0.$$

Since $\ell = 0$, from $(3.8)_2$, we see that $\tau_{ij} = 0$. □

4 Indefinite complex space forms

Definition 2 *An indefinite complex space form $\bar{M}(c)$ is a connected indefinite Kaehler manifold of constant holomorphic sectional curvature c;*

$$\begin{aligned}\widetilde{R}(\bar{X},\bar{Y})\bar{Z} \;=\; & \frac{c}{4}\{\bar{g}(\bar{Y},\bar{Z})\bar{X}-\bar{g}(\bar{X},\bar{Z})\bar{Y} \\ & +\bar{g}(J\bar{Y},\bar{Z})J\bar{X}-\bar{g}(J\bar{X},\bar{Z})J\bar{Y}+2\bar{g}(\bar{X},J\bar{Y})J\bar{Z}\},\end{aligned} \tag{4.1}$$

where $\widetilde{R}$ is the curvature tensor of the Levi-Civita connection $\widetilde{\nabla}$ on $\bar{M}$.

Denote by $\bar{R}$ the curvature tensor of the (ℓ,m)-type metric connection $\bar{\nabla}$ on $\bar{M}$. By direct calculations from (1.2) and (1.3), we see that

$$\begin{aligned}\bar{R}(\bar{X},\bar{Y})\bar{Z} \;=\; & \widetilde{R}(\bar{X},\bar{Y})\bar{Z} \\ & +(X\ell)\{\theta(Z)Y-g(Y,Z)\zeta\}-(Xm)\theta(Y)JZ \\ & -(Y\ell)\{\theta(Z)X-g(X,Z)\zeta\}+(Ym)\theta(X)JZ \\ & +\ell\{(\bar{\nabla}_X\theta)(Z)Y-(\bar{\nabla}_Y\theta)(Z)X \\ & \qquad +g(X,Z)\bar{\nabla}_Y\zeta-g(Y,Z)\bar{\nabla}_X\zeta \\ & \qquad +\ell[g(Y,Z)X-g(X,Z)Y]\} \\ & -m\{(\bar{\nabla}_X\theta)(Y)-(\bar{\nabla}_Y\theta)(X) \\ & \qquad +m[\theta(Y)\theta(JX)-\theta(X)\theta(JY)]\}JZ \\ & +\ell m\{[\theta(Y)JX-\theta(X)JY]\theta(Z) \\ & \qquad -[\theta(Y)g(JX,Z)-\theta(X)g(JY,Z)]\zeta\}.\end{aligned} \tag{4.2}$$

Applying $\bar{\nabla}_X$ to $\bar{g}(\zeta,\xi_i)=0$ and $\bar{g}(\zeta,N_i)=0$ by turns and using (2.3), (2.4), (2.7), (2.18), (2.20), and the fact that $\bar{\nabla}$ is metric, we obtain

$$\bar{g}(\bar{\nabla}_X\zeta,\xi_i)=h_i^{\ell}(X,\zeta), \qquad \bar{g}(\bar{\nabla}_X\zeta,N_i)=h_i^{*}(X,\zeta). \tag{4.3}$$

In general, applying $\bar{\nabla}_X$ to $\theta(\xi_i)=0$ and using (2.3), (2.7), (2.18), and the facts that $\theta(N_i)=\theta(E_a)=0$ we obtain

$$(\bar{\nabla}_X\theta)(\xi_i)=h_i^{\ell}(X,\zeta). \tag{4.4}$$

Denote by R and R^* the curvature tensor of the induced linear connections ∇ and ∇^* on M and $S(TM)$, respectively. Using the Gauss-Weingarten formulae, we obtain Gauss equations for M and $S(TM)$, respectively:

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z \\ &\quad + \sum_{i=1}^{r}\{h_i^\ell(X,Z)A_{N_i}Y - h_i^\ell(Y,Z)A_{N_i}X\} \\ &\quad + \sum_{a=r+1}^{n}\{h_a^s(X,Z)A_{E_a}Y - h_a^s(Y,Z)A_{E_a}X\} \\ &+ \sum_{i=1}^{r}\{(\nabla_X h_i^\ell)(Y,Z) - (\nabla_Y h_i^\ell)(X,Z) \\ &\quad + \sum_{j=1}^{r}[\tau_{ji}(X)h_j^\ell(Y,Z) - \tau_{ji}(Y)h_j^\ell(X,Z)] \\ &\quad + \sum_{a=r+1}^{n}[\lambda_{ai}(X)h_a^s(Y,Z) - \lambda_{ai}(Y)h_a^s(X,Z)] \\ &\quad - \ell[\theta(X)h_i^\ell(Y,Z) - \theta(Y)h_i^\ell(X,Z)] \\ &\quad - m[\theta(X)h_i^\ell(FY,Z) - \theta(Y)h_i^\ell(FX,Z)]\}N_i \\ &\quad + \sum_{a=r+1}^{n}\{(\nabla_X h_a^s)(Y,Z) - (\nabla_Y h_a^s)(X,Z) \\ &\quad + \sum_{i=1}^{r}[\rho_{ia}(X)h_i^\ell(Y,Z) - \rho_{ia}(Y)h_i^\ell(X,Z)] \\ &\quad + \sum_{b=r+1}^{n}[\mu_{ba}(X)h_b^s(Y,Z) - \mu_{ba}(Y)h_b^s(X,Z)] \\ &\quad - \ell[\theta(X)h_a^s(Y,Z) - \theta(Y)h_a^s(X,Z)] \\ &\quad - m[\theta(X)h_a^s(FY,Z) - \theta(Y)h_a^s(FX,Z)]\}E_a,\end{aligned} \tag{4.5}$$

$$\begin{aligned}R(X,Y)PZ &= R^*(X,Y)PZ \\ &+ \sum_{i=1}^{r}\{h_i^*(X,PZ)A^*_{\xi_i}Y - h_i^*(Y,PZ)A_{\xi_i}X\} \\ &+ \sum_{i=1}^{r}\{(\nabla_X h_i^*)(Y,PZ) - (\nabla_Y h_i^*)(X,PZ) \\ &+ \sum_{k=1}^{r}[\tau_{ik}(Y)h_k^*(X,PZ) - \tau_{ik}(X)h_k^*(Y,PZ)] \\ &- \ell[\theta(X)h_i^*(Y,PZ) - \theta(Y)h_i^*(X,PZ)] \\ &- m[\theta(X)h_i^*(FY,PZ) - \theta(Y)h_i^*(FX,PZ)]\}\xi_i.\end{aligned} \tag{4.6}$$

Taking the scalar product of N_i and (4.2) and using (4.1), (4.5), (4.6),

$(4.3)_1$, and the facts that ζ belongs to $S(TM)$ and $\bar{\nabla}$ is a metric, we obtain

$$
\begin{aligned}
&(\nabla_X h_i^*)(Y,\,PZ) - (\nabla_Y h_i^*)(X,\,PZ) \\
&- \sum_{k=1}^{r} \{\tau_{ik}(X)h_k^*(Y,PZ) - \tau_{ik}(Y)h_k^*(X,PZ)\} \\
&- \sum_{k=1}^{r} \{h_k^\ell(Y,PZ)\eta_i(A_{N_k}X) - h_k^\ell(X,PZ)\eta_i(A_{N_k}Y)\} \\
&- \sum_{a=r+1}^{n} \{h_a^s(Y,\,PZ)\eta_i(A_{E_a}X) - h_a^s(X,\,PZ)\eta_i(A_{E_a}Y)\} \\
&- \ell\{\theta(X)h_i^*(Y,PZ) - \theta(Y)h_i^*(X,PZ)\} \\
&- m\{\theta(X)h_i^*(FY,PZ) - \theta(Y)h_i^*(FX,PZ)\} \\
&- \{(X\ell)\eta_i(Y) - (Y\ell)\eta_i(X)\}\theta(PZ) \\
&+ \{(Xm)\theta(Y) - (Ym)\theta(X)\}v_i(PZ) \\
&- \ell\{(\bar{\nabla}_X\theta)(PZ)\eta_i(Y) - (\bar{\nabla}_Y\theta)(PZ)\eta_i(X)\} \\
&- \ell\{g(X,PZ)h_i^*(Y,\zeta) - g(Y,PZ)h_i^*(X,\zeta)\} \\
&- \ell^2\{g(Y,PZ)\eta_i(X) - g(X,PZ)\eta_i(Y)\} \\
&+ m\{(\bar{\nabla}_X\theta)(Y) - (\bar{\nabla}_Y\theta)(X) \\
&\qquad + m[\theta(Y)\theta(FX) - \theta(X)\theta(FY)]\}v_i(PZ) \\
&- \ell m\{\theta(Y)v_i(X) - \theta(X)v_i(Y)\}\theta(PZ) \\
&= \frac{c}{4}\{g(Y,PZ)\eta_i(X) - g(X,PZ)\eta_i(Y) + v_i(X)\bar{g}(JY,PZ) \\
&\qquad - v_i(Y)\bar{g}(JX,PZ) + 2v_i(PZ)\bar{g}(X,JY)\}.
\end{aligned}
\tag{4.7}
$$

Theorem 4 *Let M be a generic lightlike submanifold of an indefinite complex space form $\bar{M}(c)$ with an (ℓ,m)-type metric connection subject such that ζ belongs to $S(TM)$. If (1) F is parallel with respect to ∇ or (2) U_i's are parallel with respect to ∇ and $\rho_{ia}=0$, then $c=0$ and $\bar{M}(c)$ is flat.*

Proof. (1) If F is parallel with respect to the connection ∇, then by Theorem 1 $\ell=0$ and M is statical. Thus, (2.24) holds. Taking the scalar product of U_j and $(3.3)_1$ and using (2.20), we have

$$h_i^*(X,U_j)=0.$$

Applying ∇_X to $h_i^*(Y,U_j)=0$ and using $(3.4)_1$, we obtain

$$(\nabla_X h_i^*)(Y,U_j)=0.$$

Taking $PZ=U_j$ in (4.7) and using (2.24) and the above equations, we get

$$\frac{c}{4}\{\eta_i(X)v_j(Y) - \eta_i(Y)v_j(X) - \eta_j(Y)v_i(X) + \eta_j(X)v_i(Y)\}=0,$$

since $\ell = 0$. Substituting $X = \xi_i$ and $Y = V_j$ here, we obtain $c = 0$.

(2) If U_i's are parallel with respect to ∇ and $\rho_{ia} = 0$, then by Theorem 3.2 M is solenoidal and (3.5) and (3.6) hold. Further, $g(F\zeta, \zeta) = 0$ since $\bar{g}(J\zeta, \zeta) = 0$. Taking in turns the scalar product of $F\zeta$, N_j and (2.26) with $\nabla_X U_i = 0$ and using $(2.1)_2$, (2.11), (2.13), and $(3.6)_{1,2}$, we have

$$\ell h_i^*(X, \zeta) = \ell^2 \eta_i(X), \qquad h_i^*(X, U_j) = 0. \tag{4.8}$$

Applying ∇_X to $h_i^*(Y, U_j) = 0$ and using the fact that $\nabla_X U_j = 0$, we get

$$(\nabla_X h_i^*)(Y, U_j) = 0.$$

Taking $PZ = U_j$ in (4.7) and using (2.24), (3.5), (3.6), (4.8), and the last two equations, we obtain

$$\frac{c}{4}\{\eta_i(X)v_j(Y) - \eta_i(Y)v_j(X) + \eta_j(X)v_i(Y) - \eta_j(Y)v_i(X)\} = 0.$$

Substituting $X = \xi_i$ and $Y = V_j$ into this equation, we have $c = 0$. □

Theorem 5 *Let M be a solenoidal generic lightlike submanifold of an indefinite complex space form $\bar{M}(c)$ with an (ℓ, m)-type metric connection such that ζ is tangent to M. If V_i's are parallel with respect to ∇ and $\lambda_{ia} = 0$, then the function m satisfies the partial differential equation*

$$(\xi_i m)\theta(U_j) + m\{(\bar{\nabla}_{\xi_i}\theta)(U_j) - \delta_{ij}\} = \frac{3}{4}c\,\delta_{ij}. \tag{4.9}$$

Proof. If V_i's are parallel with respect to ∇ and $\lambda_{ai} = 0$, then $\ell = 0$, $\tau_{ij} = 0$ and M is irrotational. Taking $X = U_j$ in (4.4) and using (3.10), we obtain

$$(\bar{\nabla}_{U_j}\theta)(\xi_i) = m\{\delta_{ij} - \theta(U_j)\theta(V_i)\}. \tag{4.10}$$

From $(2.25)_1$, (3.7) and $(3.8)_1$, we get

$$h_i^*(X, V_k) = 0.$$

Applying ∇_X to $h_i^*(Y, V_k) = 0$ and using the fact that $\nabla_X V_k = 0$, we obtain

$$(\nabla_X h_i^*)(Y, V_k) = 0.$$

Taking $PZ = V_k$ in (4.7) and using the last two equations and the fact that $\ell = 0$, we get

$$\begin{aligned}
&\{(Xm)\theta(Y) - (Ym)\theta(X)\}\delta_{ik} \\
&+ m\{(\bar{\nabla}_X\theta)(Y) - (\bar{\nabla}_Y\theta)(X) \\
&\qquad + m[\theta(Y)\theta(FX) - \theta(X)\theta(FY)]\}\delta_{ik} \\
&= \frac{c}{4}\{u_k(Y)\eta_i(X) - u_k(X)\eta_i(Y) + 2\bar{g}(X, JY)\delta_{ik}\}.
\end{aligned}$$

Substituting $X = \xi_k$ and $Y = U_j$ into this equation and using (4.10), we see that (4.9) holds. □

Definition 3 *We say that $S(TM)$ is totally umbilical [2] in M if there exist smooth functions $\gamma_i, i \in \{1, \cdots, r\}$ on a coordinate neighborhood $\mathcal{U}$ of M such that*

$$h_i^*(X, PY) = \gamma_i g(X, PY) \qquad \text{for any } i. \tag{4.11}$$

In case $\gamma_i = 0$ on $\mathcal{U}$, we say that $S(TM)$ is totally geodesic in M.

Theorem 6 *Let M be a statical generic lightlike submanifold of an indefinite complex space form $\bar{M}(c)$ with an (ℓ, m)-type metric connection such that ζ belongs to $S(TM)$. If $S(TM)$ is totally umbilical in M, then*

$$U_k\ell - \ell^2\theta(U_k) - m\gamma_k - \gamma_i\{\gamma_i + m(U_i)\}\theta(V_k) = 0. \tag{4.12}$$

Moreover, if $S(TM)$ is totally geodesic in M, then

$$(\xi_k m)\theta(U_i) + m(\bar{\nabla}_{\xi_k}\theta)(U_i) - m^2\delta_{ki} = \frac{3}{4}c\,\delta_{ki}. \tag{4.13}$$

Proof. Since M is statical, we obtain (2.23) and (2.24). Also, since $S(TM)$ is totally umbilical, from $(2.25)_1$ and (4.11), we see that

$$h_j^\ell(X, U_i) = \gamma_i u_j(X) - \ell\theta(V_j)\eta_i(X).$$

Substituting $X = \xi_j$ into this equation and using (2.16) and $(2.23)_1$, we have

$$\ell\theta(V_i) = 0, \qquad h_j^\ell(X, U_i) = \gamma_i u_j(X), \tag{4.14}$$
$$h_j^\ell(U_i, X) = \{\gamma_i - m\theta(U_i)\}u_j(X) + m\theta(X)\delta_{ij}.$$

Replacing X by V_k and ζ in $(4.14)_3$ in turns, we obtain

$$h_j^\ell(U_i, V_k) = m\theta(V_k)\delta_{ij}, \quad h_j^\ell(U_i, \zeta) = \{\gamma_i - m\theta(U_i)\}\theta(V_j) + m\delta_{ij}. \tag{4.15}$$

Taking $X = U_j$ in (4.4) and using $(4.15)_2$, we have

$$(\bar{\nabla}_{U_i}\theta)(\xi_j) = \{\gamma_i - m\theta(U_i)\}\theta(V_j) + m\delta_{ij}. \tag{4.16}$$

Applying $\bar{\nabla}_X$ to $\ell\theta(V_i) = 0$ and using (2.18), (2.27) and $(4.14)_1$, we obtain

$$(X\ell)\theta(V_i) + \ell(\bar{\nabla}_X\theta)(V_i) = \ell\{h_i^\ell(X, F\zeta) + \ell u_i(X)\},$$

since $\lambda_{ai} = 0$. Taking $X = F\zeta$ in $(4.14)_2$ and using (2.16), we have

$$h_j^\ell(U_i, F\zeta) = 0.$$

Replacing X by U_j in the last equation, we obtain

$$(U_j\ell)\theta(V_i) + \ell(\bar{\nabla}_{U_j}\theta)(V_i) = \ell^2\delta_{ij}. \tag{4.17}$$

Applying ∇_X to $h_i^*(Y, PZ) = \gamma_i g(Y, PZ)$ and using (2.14), we obtain

$$(\nabla_X h_i^*)(Y, PZ) = (X\gamma_i)g(Y, PZ) + \gamma_i \sum_{j=1}^{r} h_j^\ell(X, PZ)\eta_j(Y).$$

Substituting this equation and (4.11) into (4.7) and using (2.24), we get

$$\begin{aligned}
&\{X\gamma_i - \sum_{j=1}^{r}\gamma_j\tau_{ij}(X)\}g(Y, PZ) - \{Y\gamma_i - \sum_{j=1}^{r}\gamma_j\tau_{ij}(Y)\}g(X, PZ)\\
&+ \gamma_i\sum_{j=1}^{r}\{h_j^\ell(X, PZ)\eta_j(Y) - h_j^\ell(Y, PZ)\eta_j(X)\}\\
&- m\gamma_i\{\theta(X)g(FY, PZ) - \theta(Y)g(FX, PZ)\}\\
&- \{\{(X\ell)\theta(PZ) + \ell(\bar\nabla_X\theta)(PZ) - \ell^2 g(X, PZ)\}\eta_i(Y)\\
&+ \{\{(Y\ell)\theta(PZ) + \ell(\bar\nabla_Y\theta)(PZ) - \ell^2 g(Y, PZ)\}\eta_i(X)\\
&+ \{(Xm)\theta(Y) - (Ym)\theta(X)\}v_i(PZ)\\
&+ m\{(\bar\nabla_X\theta)(Y) - (\bar\nabla_Y\theta)(X)\\
&\qquad + m[\theta(Y)\theta(FX) - \theta(X)\theta(FY)]\}v_i(PZ)\\
&- \ell m\{\theta(Y)v_i(X) - \theta(X)v_i(Y)\}\theta(PZ)\\
&= \frac{c}{4}\{g(Y, PZ)\eta_i(X) - g(X, PZ)\eta_i(Y)\\
&\quad + v_i(X)\bar g(JY, PZ) - v_i(Y)\bar g(JX, PZ) + 2v_i(PZ)\bar g(X, JY)\}.
\end{aligned}$$

Replacing Y by ξ_k in this equation and using (2.25), (2.9) and (2.10), we have

$$\begin{aligned}
&\{\xi_k\gamma_i - \sum_{j=1}^{r}\gamma_j\tau_{ij}(\xi_k)\}g(X, PZ) - \gamma_i h_k^\ell(X, PZ)\\
&- m\gamma_i\theta(X)u_k(PZ) + (\xi_k m)\theta(X)v_i(PZ)\\
&+ \{(X\ell)\theta(PZ) + \ell(\bar\nabla_X\theta)(PZ) - \ell^2 g(X, PZ)\}\delta_{ik}\\
&- \{(\xi_k\ell)\theta(PZ) + \ell(\bar\nabla_{\xi_k}\theta)(PZ)\}\eta_i(X)\\
&- m\{(\bar\nabla_X\theta)(\xi_k) - (\bar\nabla_{\xi_k}\theta)(X) + m\theta(X)\theta(V_k)\}v_i(PZ)\\
&= \frac{c}{4}\{g(X, PZ)\delta_{ik} + v_i(X)u_k(PZ) + 2v_i(PZ)u_k(X)\}.
\end{aligned} \tag{4.18}$$

Taking $X = U_h$, $PZ = V_h$ and using $(4.15)_1$, (4.16) and (4.17), we get

$$\begin{aligned}
&\xi_k\gamma_i - \sum_{j=1}^{r}\gamma_j\tau_{ij}(\xi_k) - 2m\gamma_i\theta(V_k)\\
&+ (\xi_k m)\theta(U_i) + m(\bar\nabla_{\xi_k}\theta)(U_i) - m^2\delta_{ik} = \frac{3}{4}c\delta_{ik}.
\end{aligned} \tag{4.19}$$

Applying $\bar{\nabla}_X$ to $\theta(\zeta) = 1$ and using the fact that $\bar{\nabla}$ is a metric, we obtain

$$(\bar{\nabla}_X \theta)(\zeta) = 0. \tag{4.20}$$

Taking $X = U_i$ and $PZ = \zeta$ in (4.18) and using $(4.15)_2$, (4.16), (4.19), and (4.20), we obtain (4.12). If (TM) is totally geodesic in M, that is, $\gamma_i = 0$, then, from (4.19), we get (4.13). □

References

[1] G. de Rham, *Sur la réductibilité d'un espace de Riemannian*, Comm. Math. Helv., **26** (1952), 328–344.

[2] K. L. Duggal and A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Kluwer Acad. Publishers, Dordrecht, 1996.

[3] K. L. Duggal and D. H. Jin, *Generic lightlike submanifolds of an indefinite Sasakian manifold,* Int. Elec. J. Geo., **5** (2012), no. 1, 108–119.

[4] H. A. Hayden, *Subspace of a space with torsion*, Proc. London Math. Soc., **34** (1932), 27–50.

[5] D. H. Jin, *Indefinite generalized Sasakian space form admitting a generic lightlike submanifold,* Bull. Korean Math. Soc., **51** (2014), no. 6, 1711–1726.

[6] D. H. Jin, *Generic lightlike submanifolds of an indefinite trans-Sasakian manifold of a quasi-constant curvature,* Appl. Math. Sci., **9** (2015), no. 60, 2985–2997.

[7] D. H. Jin, *Lightlike hypersurfaces of an indefinite generalized Sasakian space form with a symmetric metric connection of type* (ℓ, m), Commun. Korean Math. Soc., **31** (2016), no. 3, 613–624.

[8] D. H. Jin, *Lightlike hypersurfaces of an indefinite Kaehler manifold with a symmetric metric connection of type* (ℓ, m), Bull. Korean Math. Soc., **53** (2016), no. 4, 1171–1184.

[9] D. H. Jin, *Lightlike hypersurfaces of an indefinite trans-Sasakian manifold with an* $(\ell,\ m)$*-type metric connection*, Far East J. Math. Sci., **103** (2008), no. 8, 1323–1343.

[10] D. H. Jin and J. W. Lee, *Generic lightlike submanifolds of an indefinite cosymplectic manifold,* Math. Probl. in Engin., **2011** (2011), Art ID 610986, 1–16.

[11] D. H. Jin and J. W. Lee, *Generic lightlike submanifolds of an indefinite Kaehler manifold,* Inter. J. Pure and Appl. Math., **101** (2015), no. 4, 543–560.

[12] D. H. Jin and J. W. Lee, *A semi-Riemannian manifold of quasi-constant curvature admits lightlike submanifolds*, Inter. J. of Math. Analysis, **9** (2015), no. 26, 1215–1229.

[13] D. N. Kupeli, *Singular Semi-Riemannian Geometry*, Kluwer Academic, 1996.

[14] K. Yano, *On semi-symmetric metric connection*, Rev. Roum. Math. Pures et Appl., **15** (1970), 1579–1586.

[15] K. Yano and T. Imai, *Quarter-symmetric metric connection and their curvature tensors*, Tensor, N.S., **38** (1982), 13–18.

Chul Woo Lee
Department of Mathematics,
Kyungpook National University
Daegu 41566, Republic of Korea.
mathisu@knu.ac.kr

Jae Won Lee (Corresponding author)
Department of Mathematics and RINS,
Gyeongsang National University
Jinju 52828, Republic of Korea.
leejaew@gnu.ac.kr

Please, cite to this paper as published in
Armen. J. Math., V. **12**, N. 6(2020), pp. 1–17