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On the power integrability with a weight of trigonometric series from $RBVS^{r,\delta}_{+,\omega}$ class

Xh. Z. Krasniqi

Abstract. In this article, we have presented the necessary and sufficient conditions for the power integrability with a weight of the sum of sine and cosine series whose coefficients belong to the $RBVS^{r;\delta}_{+,\omega}$ class.

Key Words: L^p-integrability, Trigonometric series, Almost monotone sequence, Rest bounded variation sequence, Modulus of continuity
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Introduction

We consider the formal series

$$g(x) := \sum_{n=1}^{\infty} a_n \sin nx \tag{1}$$

and

$$f(x) := \sum_{n=1}^{\infty} b_n \cos nx. \tag{2}$$

Pertaining to the series (1)–(2), typically, the following questions are of special interest:
(q1) Are they pointwise (uniform) convergent?
(q2) Are they Fourier series of their sums?
(q3) Are they convergent in L^1-norm?
(q4) Are their formal sums L^p-integrable with a specific weight?

In this paper, we predominantly deal with problems (q1) and (q4). Solely, we touch a little bit question (q1) and we do not treat questions (q2)–(q3) at all.

Seemingly, Young [15] and Boas [1], and then Haywood [3], were the first who had studied the integrability of the series (1)–(2), imposing certain conditions on the coefficients a_n and b_n, respectively (hereinafter we denote

λ_n either a_n or b_n). They have treated trigonometric series whose coefficients are monotone decreasing. Regarding their uniform convergence, Leindler [6] replaced the monotonicity condition on the sequence $\lambda := \{\lambda_n\}$ by a more general one: $\{\lambda_n\} \in R_0^+BVS$.

Definition 1 *A sequence $c := \{c_n\}$ of positive numbers tending to zero is of rest bounded variation, or briefly $c \in R_0^+BVS$, if it possesses the property*

$$\sum_{n=m}^{\infty} |c_n - c_{n+1}| \leq K(c)c_m \tag{3}$$

for all natural numbers m, where $K(c)$ is a constant depending only on c.

Relatively lately, Németh [9] considered weight functions more general than power one and obtained some sufficient conditions for the integrability of the sine series with such weights. He used the so-called almost increasing (decreasing) sequences.

A sequence $\gamma := \{\gamma_n\}$ of positive terms will be called almost increasing (decreasing) if there exists a constant $C := C(\gamma) \geq 1$ such that

$$C\gamma_n \geq \gamma_m \quad (\gamma_n \leq C\gamma_m)$$

holds for any $n \geq m$.

Here and in the sequel, the function $\gamma(x)$ is defined by the sequence γ in the following way: $\gamma\left(\frac{\pi}{n}\right) := \gamma_n$, $n \in \mathbb{N}$, and there exist positive constants C_1 and C_2 such that $C_1\gamma_n \leq \gamma(x) \leq C_2\gamma_{n+1}$ for $x \in \left(\frac{\pi}{n+1}, \frac{\pi}{n}\right)$.

Tikhonov (see [12]) has proved two theorems providing necessary and sufficient conditions for the $p-$th power integrability of the sums of sine and cosine series with weight γ. His results refine the assertions of such results presented earlier by others and show that such conditions depend on the behavior of the sequence γ.

Tikhonov's results are the following.

Theorem 1 ([12]) *Suppose that $\{\lambda_n\} \in R_0^+BVS$ and $1 \leq p < \infty$.*

(A) If the sequence $\{\gamma_n\}$ satisfies the condition: there exists $\varepsilon_1 > 0$ such that the sequence $\{\gamma_n n^{-p-1+\varepsilon_1}\}$ is almost decreasing, then the condition

$$\sum_{n=1}^{\infty} \gamma_n n^{p-2} \lambda_n^p < \infty \tag{4}$$

is sufficient for the validity of the inclusion

$$\gamma(x)|g(x)|^p \in L(0, \pi). \tag{5}$$

(B) *If the sequence* $\{\gamma_n\}$ *satisfies the condition: there exists* $\varepsilon_2 > 0$ *such that the sequence* $\{\gamma_n n^{p-1-\varepsilon_2}\}$ *is almost increasing, then the condition (4) is necessary for the validity of (5).*

Theorem 2 ([12]) *Suppose that* $\{\lambda_n\} \in R_0^+BVS$ *and* $1 \le p < \infty$.

(A) *If the sequence* $\{\gamma_n\}$ *satisfies the condition: there exists* $\varepsilon_3 > 0$ *such that the sequence* $\{\gamma_n n^{-1+\varepsilon_3}\}$ *is almost decreasing, then the condition*

$$\sum_{n=1}^{\infty} \gamma_n n^{p-2} \lambda_n^p < \infty \tag{6}$$

is sufficient for the validity of the inclusion

$$\gamma(x)|f(x)|^p \in L(0,\pi). \tag{7}$$

(B) *If the sequence* $\{\gamma_n\}$ *satisfies the condition: there exists* $\varepsilon_4 > 0$ *such that the sequence* $\{\gamma_n n^{p-1-\varepsilon_4}\}$ *is almost increasing, then the condition (6) is necessary for the validity of (7).*

More results concerning the problems mentioned above can be found in [13], [14], [2], [11], and [4].

Very recently, the present author and Szal [5] have obtained the counterparts of Theorems 1–2 (we intentialy do not recall them) considering condition $\{\lambda_n\} \in RBVS_+^{r,\delta}$ instead of $\{\lambda_n\} \in R_0^+BVS$.

Definition 2 *A sequence* $c := \{c_k\}$ *of nonnegative numbers tending to zero belongs to* $RBVS_+^{r,\delta}$ *if it has the property*

$$\sum_{k=m}^{\infty} |c_k - c_{k+1}| \le \frac{K(c)}{m^{r+1+\delta}} \sum_{n=1}^{m} n^{r+1} c_n$$

for all natural numbers m, *where* $r \in \mathbb{N} \cup \{0\}$, $0 < \delta \le 1$ *and* $K(c)$ *is a positive constant depending only on the sequence* c.

The $RBVS_+^{r,\delta}$ class has been introduced by Leindler [7], who showed that it is a wider class than the R_0^+BVS class. In fact, if $0 < \delta \le 1$ and $c \in R_0^+BVS$, then $c \in RBVS_+^{r,\delta}$ also holds true. Indeed,

$$c_m \le m^{1-\delta} c_m \le K(c) m^{-r-1-\delta} \sum_{n=1}^{m} n^{r+1} c_n.$$

Subsequently, the embedding relation $R_0^+BVS \subset RBVS_+^{r,\delta}$ holds true. Moreover, we showed in [5], that

$$RBVS_+^{r,\delta_1} \subseteq RBVS_+^{r,\delta_2} \quad (0 < \delta_2 \le \delta_1 \le 1)$$

and

$$RBVS_{+}^{r_1,\delta} \subseteq RBVS_{+}^{r_2,\delta} \quad (0 \leq r_2 \leq r_1;\ r_1, r_2 \in \mathbb{N} \cup \{0\})$$

hold true as well.

Let $\omega(\delta)$ be a non-negative and non-decreasing continuous function on $[0, 2\pi]$ having the following properties:

(i) $\omega(0) = 0$,

(ii) $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$ for $0 < \delta_1 \leq \delta_2 \leq \delta_1 + \delta_2 \leq 2\pi$.

Then $\omega(\delta)$ is called a function of modulus of continuity type.

Now we are ready to recall the $RBVS_{+,\omega}^{r,\delta}$ class of sequences, also introduced in [7].

Definition 3 *A sequence $c := \{c_k\}$ of nonnegative numbers tending to zero belongs to $RBVS_{+,\omega}^{r,\delta}$ if it has the property*

$$\sum_{k=m}^{\infty} |c_k - c_{k+1}| \leq K(c) \frac{\omega(1/m)}{m^{r+\delta}}$$

for all natural numbers m, where $r \in \mathbb{N} \cup \{0\}$, $0 < \delta \leq 1$ and $K(c)$ is a positive constant depending only on the sequence c.

Definition 4 *A sequence $c = \{c_k\}$ is said to be quasimonotone decreasing, for short we write $c \in QMS$, if*

$$0 < c_n \leq K(c) c_m \quad \text{for any} \quad n \geq m,$$

where $K(c)$ is a positive constant depending only on the sequence c.

The embedding relations

$$RBVS_0^{+} \subset QMS \subset RBVS_{+}^{r,\delta}$$

hold true, but a similar relation with $RBVS_{+,\omega}^{r,\delta}$ in place of $RBVS_{+}^{r,\delta}$, in general, does not (see [7], page 622).

Despite this, we are concerned with finding the necessary and sufficient conditions on the sequence $\{\lambda_n\} \in RBVS_{+,\omega}^{r,\delta}$ such that $\gamma(x)|g(x)|^p$, $\gamma(x)(\omega(x)|g(x)|)^p$, $\gamma(x)|f(x)|^p$, and $\gamma(x)\left(\frac{\omega(x)|f(x)|}{x}\right)^p$ belong to $L(0, \pi)$, which indeed is the aim of this paper.

1 Lemmas

The following lemmas will be applied in the proofs of the main results.

Lemma 1 ([8]) *Let $f_n > 0$ and $\mu_n \geq 0$. Then*

$$\sum_{n=1}^{\infty} f_n \left(\sum_{\nu=1}^{n} \mu_\nu\right)^p \leq p^p \sum_{n=1}^{\infty} f_n^{1-p} \mu_n^p \left(\sum_{\nu=n}^{\infty} f_\nu\right)^p, \quad p \geq 1.$$

Lemma 2 ([10]) *Let $f_n > 0$ and $\mu_n \geq 0$. Then*

$$\sum_{n=1}^{\infty} f_n \left(\sum_{\nu=n}^{\infty} \mu_\nu\right)^p \leq p^p \sum_{n=1}^{\infty} f_n^{1-p} \mu_n^p \left(\sum_{\nu=1}^{n} f_\nu\right)^p, \quad p \geq 1.$$

Now, we pass to the main results of the paper.

2 Main Results

First, we prove the following

Theorem 3 *The following statements hold true:*

(i) For any two integers r_1, r_2 such that $0 \leq r_1 \leq r_2$ and $0 < \delta \leq 1$, the embedding relation $RBVS_{+,\omega}^{r_2,\delta} \subseteq RBVS_{+,\omega}^{r_1,\delta}$ holds true.

(ii) For any integer $r \geq 0$ and $0 < \delta_1 \leq \delta_2 \leq 1$, the embedding relation $RBVS_{+,\omega}^{r,\delta_2} \subseteq RBVS_{+,\omega}^{r,\delta_1}$ holds true.

(iii) Let r be any non-negative integer and $0 < \delta \leq 1$. For any two moduli of continuity $\omega_1(t)$ and $\omega_2(t)$ such that $\omega_1(t) \leq \omega_2(t)$ for $t \in [0, 2\pi]$, the embedding relation $RBVS_{+,\omega_1}^{r,\delta} \subseteq RBVS_{+,\omega_2}^{r,\delta}$ holds true.

(iv) If $\{a_n\}, \{b_n\} \in RBVS_{+,\omega}^{r,\delta}$ with $r \geq 0$, then the functions $g(x)$ and $f(x)$ exist on $(0, \pi]$.

Proof. (i) For any two integers r_1, r_2 such that $0 \leq r_1 \leq r_2$ and $0 < \delta \leq 1$, we have

$$m^{-r_2-\delta} \leq m^{-r_2-\delta} m^{r_2-r_1} \leq m^{-r_1-\delta},$$

which proves (i).

(ii) Similarly, for $0 < \delta_1 \leq \delta_2 \leq 1$ and any integer $r \geq 0$, we write

$$m^{-r-\delta_2} \leq m^{-r-\delta_2} m^{\delta_2-\delta_1} \leq m^{-r-\delta_1},$$

which implies (ii).

(iii) The proof is obvious.

(iv) Applying summation by parts, we obtain

$$S_n^s(x) := \sum_{k=1}^{n-1} (a_k - a_{k+1}) \widetilde{D}_k(x) + a_n \widetilde{D}_n(x),$$

where $\widetilde{D}_k(x) = \sum_{j=1}^{k} \sin jx$ is the conjugate Dirichlet kernel.

Since $\{a_n\} \in RBVS^{r,\delta}_{+,\omega}$ and $r \geq 0$,

$$a_n \leq K(\lambda) n^{-r-\delta}\omega\left(\frac{1}{n}\right) \leq K(\lambda)\omega\left(\frac{1}{n}\right),$$

and hence, $a_n \to 0$ as $n \to \infty$. Therefore, taking into account that $\left|\widetilde{D}_k(x)\right| \leq Cx^{-1}$ for $x \in (0,\pi]$, we conclude that the limit

$$\lim_{n\to\infty} S_n^s(x) = \sum_{k=1}^{\infty}(a_k - a_{k+1})\widetilde{D}_k(x)$$

exists on $(0,\pi]$.

By similar arguments, we obtain the equality

$$S_n^c(x) := \sum_{k=1}^{n-1}(b_k - b_{k+1})\left(-\frac{1}{2} + D_k(x)\right) + b_n\left(-\frac{1}{2} + D_n(x)\right),$$

where $D_k(x) = \frac{1}{2} + \sum_{j=1}^{k} \cos jx$ is the Dirichlet kernel.

Since $|D_k(x)| \leq Cx^{-1}$ for $x \in (0,\pi]$, $\{b_n\} \in RBVS^{r,\delta}_{+,\omega}$, $r \geq 0$, and $b_n \to 0$ as $n \to \infty$, the limit

$$\lim_{n\to\infty} S_n^c(x) = -\frac{b_1}{2} + \sum_{k=1}^{\infty}(b_k - b_{k+1})D_k(x)$$

exists on $(0,\pi]$.

The proof is completed. □

Theorem 4 *Suppose that* $\{\lambda_n\} \in RBVS^{r,\delta}_{+,\omega}$, $r \geq 0$, $0 < \delta \leq 1$ *and* $1 \leq p < \infty$. *If the sequence* $\{\gamma_n\}$ *satisfies the condition: there exists* $\varepsilon_1 > 0$ *such that the sequence* $\{\gamma_n n^{\varepsilon_1 - 1 - \delta p}\}$ *is almost decreasing, then the condition*

$$\sum_{n=1}^{\infty} \gamma_n n^{p(1-\delta)-2}\omega^p\left(1/n\right) < \infty$$

is sufficient for the validity of the inclusion

$$\gamma\left(x\right)|g\left(x\right)|^p \in L(0,\pi).$$

Proof. Since $\lambda_n \to 0$, the use of summation by parts implies

$$\sum_{n=m+1}^{\infty} \lambda_n \sin nx = \sum_{n=m+1}^{\infty}(\lambda_n - \lambda_{n+1})\widetilde{D}_n(x) - \sum_{n=m+1}^{\infty}(\lambda_n - \lambda_{n+1})\widetilde{D}_m(x).$$

Thus, for $x \in \left(\frac{\pi}{n+1}, \frac{\pi}{n}\right]$, $|\sin nx| \le nx$, and $|\widetilde{D}_n(x)| \le Cx^{-1}$, we have

$$|g(x)| \le C\left(x\sum_{k=1}^{n} k\lambda_k + n\sum_{k=n}^{\infty}|\lambda_k - \lambda_{k+1}|\right).$$

By assumption, $\{\lambda_n\} \in RBVS^{r,\delta}_{+,\omega}$, and therefore

$$\lambda_n \le \sum_{k=n}^{\infty}|\lambda_k - \lambda_{k+1}| \le K(\lambda)n^{-r-\delta}\omega\left(\frac{1}{n}\right).$$

Using the last inequality and the properties of the modulus of continuity, we get

$$\begin{aligned}
|g(x)| &\le K(\lambda)\left(\frac{1}{n}\sum_{k=1}^{n} k\lambda_k + n^{-r+1-\delta}\omega\left(1/n\right)\right)\\
&\le K(\lambda)\left(\frac{1}{n}\sum_{k=1}^{n} k^{-r+1-\delta}\omega\left(1/k\right) + \frac{1}{n^{r+\delta}}\omega\left(1/n\right)\sum_{k=1}^{n}1\right)\\
&\le K(\lambda)\left(\frac{n^{1-\delta}}{n}\sum_{k=1}^{n} k^{-r}\omega\left(1/k\right) + \frac{1}{n^{\delta+r}}\sum_{k=1}^{n}\omega\left(1/k\right)\right)\\
&\le \frac{K(\lambda)}{n^{\delta}}\sum_{k=1}^{n} k^{-r}\omega\left(1/k\right)\\
&\le \frac{K(\lambda)}{n^{\delta}}\sum_{k=1}^{n}\omega\left(1/k\right),
\end{aligned}$$

where $K(\lambda)$ denotes a positive constant, not necessarily the same in each inequality.

Whence, we can write

$$\begin{aligned}
\int_0^{\pi}\gamma(x)|g(x)|^p dx &= \sum_{n=1}^{\infty}\int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}}\gamma(x)|g(x)|^p dx\\
&\le K(\lambda)\sum_{n=1}^{\infty}\frac{\gamma_n}{n^{2+\delta p}}\left(\sum_{k=1}^{n}\omega\left(1/k\right)\right)^p.
\end{aligned}$$

The use of Lemma 1 implies

$$\int_0^{\pi}\gamma(x)|g(x)|^p dx \le K(\lambda)\sum_{n=1}^{\infty}\left(\frac{\gamma_n}{n^{2+\delta p}}\right)^{1-p}\left(\omega\left(1/n\right)\right)^p\left(\sum_{k=n}^{\infty}\frac{\gamma_k}{k^{2+\delta p}}\right)^p.$$

Since $\{m^{\varepsilon_1-\delta p-1}\gamma_m\}$ is almost decreasing sequence, we get

$$\begin{aligned}\sum_{k=n}^{\infty}\frac{\gamma_k}{k^{2+\delta p}} &= \sum_{k=n}^{\infty}\frac{\gamma_k}{k^{1+\delta p-\varepsilon_1}k^{1+\varepsilon_1}}\\ &\leq C\frac{\gamma_n}{n^{1+\delta p-\varepsilon_1}}\sum_{k=n}^{\infty}\frac{1}{k^{1+\varepsilon_1}}\leq C\frac{\gamma_n}{n^{1+\delta p}}.\end{aligned}$$

Consequently, we obtain

$$\int_0^{\pi}\gamma(x)|g(x)|^p dx\leq K(\lambda)\sum_{n=1}^{\infty}\gamma_n n^{p(1-\delta)-2}\omega^p\left(1/n\right)<\infty.$$

The proof is completed. □

Theorem 5 *Suppose that* $\{\lambda_n\}\in RBVS^{r,\delta}_{+,\omega}$, $r\geq 0$, $0<\delta\leq 1$ *and* $1\leq p<\infty$. *If the sequence* $\{\gamma_n\}$ *satisfies the condition: there exists* $\varepsilon_2>0$ *such that the sequence* $\{\gamma_n n^{p-1-\varepsilon_2}\omega^p\left(1/n\right)\}$ *is almost increasing, then the condition*

$$\text{hl}:=\sum_{n=1}^{\infty}\gamma_n n^{p\delta-2}\lambda_n^{2p}<\infty$$

is necessary for the validity of inclusion

$$\gamma\left(x\right)\left(\omega\left(x\right)\left|g\left(x\right)\right|\right)^p\in L(0,\pi).$$

Proof. First, let us show that $g\left(x\right)\in L(0,\pi)$. Let $1<p<\infty$ and $p+q=pq$. Then the use of Hölder's inequality implies

$$\int_0^{\pi}|g(x)|dx\leq\left(\int_0^{\pi}\gamma(x)|g(x)|^p dx\right)^{1/p}\left(\int_0^{\pi}(\gamma(x))^{-q/p}dx\right)^{1/q}.$$

Using the estimate (see [12], page 440)

$$\int_0^{\pi}(\gamma(x))^{-q/p}dx<C,$$

we ensure that

$$\int_0^{\pi}|g(x)|dx\leq C\left(\int_0^{\pi}\gamma(x)|g(x)|^p dx\right)^{1/p}<\infty.$$

We consider the case $p=1$. The sequence $\{\gamma_n\}$ can be set up to be almost increasing, and consequently

$$\begin{aligned}\int_0^{\pi}|g(x)|dx &\leq \sum_{n=1}^{\infty}\frac{1}{C\gamma_n}\int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}}\gamma(x)|g(x)|dx\\ &\leq \frac{1}{C\gamma_1}\int_0^{\pi}\gamma(x)|g(x)|dx<\infty.\end{aligned}$$

Hence, for all $p \in [1, \infty)$, $g(x) \in L(0, \pi)$. This enables to integrate the function $g(x)$, which gives

$$P(x) := \int_0^x g(t)dt = \sum_{n=1}^{\infty} \lambda_n \int_0^x \sin ntdt = 2\sum_{n=1}^{\infty} \frac{\lambda_n}{n} \sin^2 \frac{nx}{2}.$$

Writing in short

$$d_r := \int_{\frac{\pi}{r+1}}^{\frac{\pi}{r}} |g(x)|dx, \quad r \in \mathbb{N},$$

taking into account that $\{\lambda_n\} \in RBVS_{+,\omega}^{r,\delta}$, and

$$\begin{aligned} P(\pi/m) &\geq C\sum_{n=1}^{m} \frac{\lambda_n}{n}\left(\frac{n}{m}\right)^2 \\ &= \frac{C}{m^2}\sum_{n=1}^{m} n\lambda_n \\ &\geq \frac{1}{K(\lambda)m^2}\sum_{n=1}^{m} n^{r+\delta+1}\lambda_n\omega^{-1}(1/n)\sum_{k=n}^{\infty}|\lambda_k-\lambda_{k+1}| \\ &\geq \frac{1}{K(\lambda)m^{r+2}}\sum_{n=1}^{m} n^{r+\delta+1}\lambda_n\omega^{-1}(1/n)\sum_{k=n}^{\infty}|\lambda_k-\lambda_{k+1}| \\ &\geq \frac{1}{K(\lambda)m^{r+2}} m^{r+\delta+1}\lambda_m\omega^{-1}(1/m)\sum_{k=m}^{\infty}|\lambda_k-\lambda_{k+1}| \\ &\geq \frac{1}{K(\lambda)} m^{\delta-1}\lambda_m^2\omega^{-1}(1/m), \end{aligned}$$

we get

$$\lambda_n^2 \leq K(\lambda)n^{1-\delta}\omega(1/n)P(\pi/n) \leq K(\lambda)n^{1-\delta}\omega(1/n)\sum_{\nu=n}^{\infty} d_\nu.$$

Thus, we have

$$\mathrm{hl} = \sum_{n=1}^{\infty}\gamma_n n^{p\delta-2}\lambda_n^{2p} \leq K(\lambda)\sum_{n=1}^{\infty}\gamma_n n^{p-2}\omega^p(1/n)\left(\sum_{\nu=n}^{\infty} d_\nu\right)^p.$$

The use of Lemma 2 gives

$$\mathrm{hl} \leq K(\lambda)\sum_{n=1}^{\infty}(d_n)^p\left(\gamma_n n^{p-2}\omega^p(1/n)\right)^{1-p}\left(\sum_{\nu=1}^{n}\gamma_\nu \nu^{p-2}\omega^p(1/\nu)\right)^p.$$

By assumption, the sequence $\{\gamma_n n^{p-1-\varepsilon_2}\omega^p(1/n)\}$ is almost increasing, which gives

$$\begin{aligned}
\text{hl} &\leq K(\lambda)\sum_{n=1}^{\infty} d_n^p \left(\gamma_n n^{p-2}\omega^p(1/n)\right)^{1-p}\left(\sum_{\nu=1}^{n}\frac{\gamma_\nu \nu^{p-1-\varepsilon_2}\omega^p(1/\nu)}{\nu^{1-\varepsilon_2}}\right)^p \\
&\leq K(\lambda)\sum_{n=1}^{\infty} d_n^p \left(\gamma_n n^{p-2}\omega^p(1/n)\right)^{1-p}\left(\gamma_n n^{p-1-\varepsilon_2}\omega^p(1/n)\sum_{\nu=1}^{n}\frac{1}{\nu^{1-\varepsilon_2}}\right)^p \\
&\leq K(\lambda)\sum_{n=1}^{\infty} d_n^p \gamma_n^{1-p} n^{(p-2)(1-p)}\omega^{p(1-p)}(1/n)\left(\gamma_n n^{p-1}\omega^p(1/n)\right)^p \\
&\leq K(\lambda)\sum_{n=1}^{\infty} d_n^p \gamma_n n^{2(p-1)}\omega^p(1/n).
\end{aligned}$$

Let $1 < p < +\infty$ and $q = p/(p-1)$. Applying Hölder's inequality, we conclude that

$$d_n^p = \left(\int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} |g(x)|dx\right)^p \leq Cn^{2(1-p)}\int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} |g(x)|^p dx.$$

Subsequently, we obtain

$$\begin{aligned}
\text{hl} &\leq K(\lambda)\sum_{n=1}^{\infty}\gamma_n\omega^p(1/n)\int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} |g(x)|^p dx \\
&\leq K(\lambda)\sum_{n=1}^{\infty}\int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} \omega^p(x)\gamma(x)|g(x)|^p dx \leq K(\lambda)\int_0^{\pi}\omega^p(x)\gamma(x)|g(x)|^p dx.
\end{aligned}$$

For $p = 1$, we conclude

$$\text{hl} \leq K(\lambda)\sum_{n=1}^{\infty}\gamma_n d_n\omega(1/n) \leq K(\lambda)\int_0^{\pi}\omega(x)\gamma(x)|g(x)|dx < \infty$$

as well.

The proof is completed. □

Theorem 6 *Suppose that* $\{\lambda_n\} \in RBVS_{+,\omega}^{r,\delta}$, $r \geq 0$, $0 < \delta \leq 1$ *and* $1 \leq p < \infty$. *If the sequence* $\{\gamma_n\}$ *satisfies the condition: there exists* $\varepsilon_3 > 0$ *such that the sequence* $\{\gamma_n n^{\varepsilon_3-1}\}$ *is almost decreasing, then the condition*

$$\sum_{n=1}^{\infty}\gamma_n n^{p(2-\delta)-2}\omega^p(1/n) < \infty,$$

is sufficient for the validity of the inclusion

$$\gamma(x)\,|f(x)|^p \in L(0,\pi).$$

Proof. We will apply the same reasoning as in the proof of Theorem 4. We have

$$\begin{aligned} |f(x)| &\le \left|\sum_{k=1}^{n} \lambda_k \cos kx\right| + \left|\sum_{k=n+1}^{\infty} \lambda_k \cos kx\right| \\ &\le \sum_{k=1}^{n} \lambda_k + \sum_{k=n}^{\infty} |\lambda_k - \lambda_{k+1}| \, |D_k(x)| + \lambda_{n+1} \, |D_n(x)| , \end{aligned}$$

where $D_n(x) = \sum_{k=1}^{n} \cos kx$, $n \in \mathbb{N}$. Hence, for $x \in \left(\frac{\pi}{n+1}, \frac{\pi}{n}\right]$, $|D_n(x)| \le Cx^{-1}$ and $\{\lambda_n\} \in RBVS_{+,\omega}^{r,\delta}$, that is,

$$\lambda_n \le \sum_{k=n}^{\infty} |\lambda_k - \lambda_{k+1}| \le K(\lambda) n^{-r-\delta} \omega\left(1/n\right),$$

we obtain

$$\begin{aligned} |f(x)| &\le C\left(\sum_{k=1}^{n} \lambda_k + n \sum_{k=n}^{\infty} |\lambda_k - \lambda_{k+1}|\right) \\ &\le K(\lambda)\left(\sum_{k=1}^{n} k^{-r-\delta}\omega\left(1/k\right) + n^{-r-\delta+1}\omega\left(1/n\right)\right) \\ &\le K(\lambda)\left(\sum_{k=1}^{n} k^{-r-\delta}\omega\left(1/k\right)\right) \le K(\lambda) \sum_{k=1}^{n} k^{1-\delta}\omega\left(1/k\right). \end{aligned}$$

Ergo,

$$\int_0^{\pi} \gamma(x)|f(x)|^p dx = \sum_{n=1}^{\infty} \int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} \gamma(x)|f(x)|^p dx \le K(\lambda) \sum_{n=1}^{\infty} \frac{\gamma_n}{n^2} \left(\sum_{k=1}^{n} k^{1-\delta}\omega\left(1/k\right)\right)^p .$$

The use of Lemma 1 and the fact that $\{m^{\varepsilon_3 - 1}\gamma_m\}$ is almost decreasing sequence, imply

$$\begin{aligned} \int_0^{\pi} \gamma(x)|f(x)|^p dx &\le K(\lambda) \sum_{n=1}^{\infty} \left(\frac{\gamma_n}{n^2}\right)^{1-p} \left(n^{1-\delta}\omega\left(1/n\right)\right)^p \left(\sum_{k=n}^{\infty} \frac{\gamma_k}{k^2}\right)^p \\ &\le K(\lambda) \sum_{n=1}^{\infty} \left(\frac{\gamma_n}{n^2}\right)^{1-p} \left(n^{1-\delta}\omega\left(1/n\right)\right)^p \left(\sum_{k=n}^{\infty} \frac{\gamma_k k^{\varepsilon_3-1}}{k^{1+\varepsilon_3}}\right)^p \\ &\le K(\lambda) \sum_{n=1}^{\infty} \left(\frac{\gamma_n}{n^2}\right)^{1-p} \left(n^{1-\delta}\omega\left(1/n\right)\right)^p \left(\gamma_n n^{\varepsilon_3-1} \sum_{k=n}^{\infty} \frac{1}{k^{1+\varepsilon_3}}\right)^p \\ &\le K(\lambda) \sum_{n=1}^{\infty} \left(\frac{\gamma_n}{n^2}\right)^{1-p} \left(n^{1-\delta}\omega\left(1/n\right)\right)^p \left(\gamma_n n^{-1}\right)^p \\ &\le K(\lambda) \sum_{n=1}^{\infty} \gamma_n n^{p(2-\delta)-2} \omega^p\left(1/n\right). \end{aligned}$$

The proof is completed. □

Theorem 7 *Suppose that* $\{\lambda_n\} \in RBVS^{r,\delta}_{+,\omega}$, $r \geq 0$, $0 < \delta \leq 1$ *and* $1 \leq p < \infty$. *If the sequence* $\{\gamma_n\}$ *satisfies the condition: there exists* $\varepsilon_2 > 0$ *such that the sequence* $\{\gamma_n n^{2p-1-\varepsilon_2}\omega^p(\pi/n)\}$ *is almost increasing, then the condition* $\mathrm{hl} < \infty$ *is necessary for the validity of inclusion*

$$\gamma(x)\left(\frac{\omega(x)|f(x)|}{x}\right)^p \in L(0,\pi).$$

Proof. Almost the same arguments, as in the proof of Theorem 5, can be used to verify that the condition $f(x) \in L(0,\pi)$ is implication of $\mathrm{hl} < \infty$. Now, integrating the function f, we get

$$Q(x) = \int_0^x f(u)\,du = \sum_{n=1}^{\infty} \frac{\lambda_n}{n} \sin nx.$$

Further, we verify that $\{\lambda_j\} \in RBVS^{r,\delta}_{+,\omega}$ implies $\{\frac{\lambda_j}{j}\} \in RBVS^{r,\delta}_{+,\omega}$. Note that $\{\lambda_j\} \in RBVS^{r,\delta}_{+,\omega}$ implies

$$\lambda_k \leq \sum_{j=k}^{\infty} |\lambda_j - \lambda_{j+1}| \leq K(\lambda) k^{-r-\delta}\omega(1/k),$$

and whence, for $m \in \mathbb{N}$, we obtain

$$\begin{aligned}
\sum_{k=m}^{\infty}\left|\frac{\lambda_k}{k} - \frac{\lambda_{k+1}}{k+1}\right| &\leq \sum_{k=m}^{\infty}\frac{1}{k}|\lambda_k - \lambda_{k+1}| + \sum_{k=m}^{\infty}\frac{1}{k(k+1)}\lambda_{k+1} \\
&\leq \frac{1}{m}\sum_{k=m}^{\infty}|\lambda_k - \lambda_{k+1}| + \sum_{k=m}^{\infty}\frac{1}{k(k+1)}\sum_{l=k+1}^{\infty}|\lambda_l - \lambda_{l+1}| \\
&\leq K(\lambda)\, m^{-r-\delta-1}\omega(1/m) + \sum_{l=m}^{\infty}|\lambda_l - \lambda_{l+1}|\sum_{k=m}^{\infty}\frac{1}{k(k+1)} \\
&\leq K(\lambda)\, m^{-r-\delta-1}\omega(1/m) \\
&\leq K(\lambda)\, m^{-r-\delta}\omega(1/m),
\end{aligned}$$

which shows that $\{\lambda_j/j\} \in RBVS^{r,\delta}_{+,\omega}$.

Applying Theorem 5 to the function Q we find that

$$\sum_{n=1}^{\infty}\gamma_n^* n^{p\delta-2}\left(\frac{\lambda_n}{n}\right)^{2p} \leq C\int_0^{\pi}\omega^p(x)\gamma^*(x)|Q(x)|^p\,dx,$$

where $\{\gamma_n^*\}$ satisfies the following condition: there exists $\varepsilon > 0$ such that the sequence $\{\gamma_n^* n^{2p-1-\varepsilon_2}\omega^p(\pi/n)\}$ is almost increasing. For $\gamma_n^* = \gamma_n n^{2p}$, this condition is satisfied as well. Then, we can write

$$\begin{aligned}
\mathrm{hl} &= \sum_{n=1}^{\infty} \gamma_n n^{p\delta-2}\lambda_n^{2p} = \sum_{n=1}^{\infty} \gamma_n^{*} n^{p\delta-2}\left(\frac{\lambda_n}{n}\right)^{2p} \\
&\leq K(\lambda)\int\limits_0^{\pi} \omega^p(x)\frac{\gamma(x)}{x^{2p}}\left|Q(x)\right|^p dx \\
&\leq K(\lambda)\sum_{n=1}^{\infty}\int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} \frac{\omega^p(x)\gamma(x)}{x^{2p}}\left(\int\limits_0^{x}\left|f(u)\right|du\right)^p dx \\
&\leq K(\lambda)\sum_{n=1}^{\infty}\gamma_n n^{2p-2}\omega^p(\pi/n)\left(\int\limits_0^{\frac{\pi}{n}}\left|f(u)\right|du\right)^p \\
&= K(\lambda)\sum_{n=1}^{\infty}\gamma_n n^{2p-2}\omega^p(\pi/n)\left(\sum_{m=n}^{\infty} h_m\right)^p,
\end{aligned}$$

where

$$h_m = \int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}} \left|f(u)\right|du, \quad m \in \mathbb{N}.$$

Now, the use of Lemma 2 implies

$$\mathrm{hl} \leq K(\lambda)\sum_{n=1}^{\infty}\gamma_n n^{3p-2}\omega^p(\pi/n)h_n^p.$$

If $1 < p < \infty$ and $q = p/(p-1)$, applying Hölder's inequality, we obtain

$$h_m^p \leq \left(\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}} 1^q\right)^{p/q}\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}} |f(x)|^p dx \leq Cm^{2(1-p)}\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}} |f(x)|^p dx.$$

Subsequently, we have

$$\begin{aligned}
\mathrm{hl} &\leq K(\lambda)\sum_{n=1}^{\infty}\gamma_n n^p\omega^p(\pi/n)\int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} |f(x)|^p dx \\
&\leq K(\lambda)\sum_{n=1}^{\infty}\int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} \frac{\omega^p(x)\gamma(x)|f(x)|^p}{x^p}dx \\
&\leq K(\lambda)\int_0^{\pi}\gamma(x)\left(\frac{\omega(x)|f(x)|}{x}\right)^p dx < \infty.
\end{aligned}$$

For $p = 1$, we also have

$$\mathrm{hl} \leq K(\lambda)\sum_{n=1}^{\infty}\gamma_n n\omega(\pi/n)h_n \leq K(\lambda)\int_0^{\pi}\frac{\omega(x)\gamma(x)|f(x)|}{x}dx < \infty,$$

and the proof is completed. □

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Xhevat Z. Krasniqi
Department of Mathematics and Informatics,
Faculty of Education, University of Prishtina
Mother Theresa ave., no. 5, 10000 Prishtina, Kosovo
xhevat.krasniqi@uni-pr.edu

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