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Hyperidentities with permutations in invertible binary algebras

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Abstract. In this paper, using hyperidentities with permutations we obtained characterization invertible algebras of various types of linearity.

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Introduction

A binary algebra $(Q;\Sigma)$ is called invertible algebra or system of quasigroups if each operation in Σ is a quasigroup operation. Invertible algebras with second-order formulas first were considered by Shaufler [17] in connection with coding theory. He pointed out that the resulting message would be more difficult to decode by an unauthorized receiver than in the case when a single operation is used for calculation. Later such algebras were investigated by J. Aczel [1], V. D. Belousov [3, 5], A. Sade [16], Yu. M. Movsisyan [11]–[15] and others.

It is well known [4] that with each quasigroup A the next five quasigroups are connected:

$$A^{-1},\ \ {}^{-1}A,\ \ {}^{-1}(A^{-1}),\ \ ({}^{-1}A)^{-1},\ \ A^{*},$$

where $A^{*}(x,y)=A(y,x)$. These quasigroups are called parastrophes of a quasigroup A. Like this, with each invertible algebra $(Q;\Sigma)$ the next five invertible algebras are connected:

$$(Q;\Sigma^{-1}),\ \ (Q;{}^{-1}\Sigma),\ \ (Q;{}^{-1}(\Sigma^{-1})),\ \ (Q;({}^{-1}\Sigma)^{-1}),\ \ (Q;\Sigma^{*}),$$

where

$$\Sigma^{-1}=\{A^{-1}|\,A\in\Sigma\},$$
$${}^{-1}\Sigma=\{{}^{-1}A|\,A\in\Sigma\},$$
$${}^{-1}(\Sigma^{-1})=\{{}^{-1}(A^{-1})|\,A\in\Sigma\},$$
$$({}^{-1}\Sigma)^{-1}=\{({}^{-1}A)^{-1}|\,A\in\Sigma\},$$
$$\Sigma^{*}=\{A^{*}|\,A\in\Sigma\}.$$

Each of these invertible algebras are called parastrophes of the algebra $(Q;\Sigma)$. The algebra $(Q;\Sigma^*)$ is called dual algebra of the $(Q;\Sigma)$.

Let us recall that the following absolutely closed second-order formulas:

$$\forall X_1,\dots,X_m \forall x_1,\dots,x_n \quad (\omega_1=\omega_2),$$
$$\forall X_1,\dots,X_k \exists X_{k+1}\dots,X_m \forall x_1,\dots,x_n \quad (\omega_1=\omega_2),$$

where ω_1, ω_2 are words written in the functional variables $X_1,\dots,X_m$ and in the object variables $x_1,\dots,x_n$, are called $\forall(\forall)$-identity or hyperidentity and $\forall\exists(\forall)$-identity, respectively. For details see [12].

The notion of isotopy plays an important role in the theory of quasigroups. The groupoid $(Q;A)$ is called isotopic to the groupoid $(Q;B)$ if exist three maps α,β,γ of Q to Q such that $\gamma B(x,y)=A(\alpha x,\beta y)$ for all $x,y\in Q$. The isotopy of the form $T=(\alpha,\beta,\varepsilon)$, where ε is the identity map, is called a principal isotope.

Many authors studied quasigroups isotopic to the groups or their different subclasses and identities that generates them [18]-[21]. In [5], Belousov introduced the concept of linear and left (right) linear quasigroups. Later linear and left (right) linear invertible algebras were defined, which were characterized by second-order formulas [7, 8, 13]. The notion of identity with permutations first has been introduced in [6]. In [9], we brought the notion of hyperidentity with permutations, with which we characterized hyperidentities bringing to the isotopy of quasigroups to groups.

In this paper, we obtain characterizations of invertible algebras with different types of linearity using hyperidentities with permutations.

1 Auxiliary concepts and results

In this section, we present some notions and results, which are necessary for further considerations.

Definition 1 *The triple $T=(\alpha,\beta,\gamma)$ of permutations of the set Q is called an autotopy of the groupoid $(Q;\cdot)$ if the following identity*

$$\gamma(x\cdot y)=\alpha x\cdot\beta y$$

is true for $\forall x,y\in Q$.

Consequently, the notion of autotopy is the particular case of the notion of isotopy. It is sufficient to state that $A=B$. If $T=(\alpha,\beta,\gamma)$ is an autotopy of the groupoid $(Q;A)$, we write $A^T=A$. In case $\alpha=\beta=\gamma$, the triple $T=(\alpha,\alpha,\alpha)$ is an automorphism of the groupoid $(Q;\cdot)$. It is easy to see that the set of autotopies of $(Q;\cdot)$ forms a group.

Definition 2 *The third component γ of the autotopy $T = (\alpha, \beta, \gamma)$ of the groupoid $(Q; \cdot)$ is called quasi-automorphism of the groupoid $(Q; \cdot)$. In other words, a permutation $\gamma : Q \to Q$ is called quasi-automorphism of the groupoid $(Q; \cdot)$ if there exists such a permutations $\alpha, \beta : Q \to Q$ that*

$$\gamma(x \cdot y) = \alpha x \cdot \beta y,$$

i.e., $T = (\alpha, \beta, \gamma)$ is an autotopy of the groupoid $(Q; \cdot)$.

In case of the groups, quasi-automorphisms have an easy construction.

Lemma 1 *[5] Let α, β, γ, δ, σ, τ be permutations of the set Q such that the equality*

$$\beta(\alpha(x \cdot y) \cdot z) = \gamma x \cdot \delta(\sigma y \cdot \tau z)$$

is valid in the group $(Q; \cdot)$ for all $x, y, z \in Q$. Then the permutations α, β, γ, δ, σ, τ are quasi-automorphisms of the group $(Q; \cdot)$.

Definition 3 *An invertible algebra $(Q; \Sigma)$ is called left (right) linear over a group $(Q; +)$ if every operation $A \in \Sigma$ has the following form:*

$$A(x, y) = \varphi_A x + \beta_A y \quad (A(x, y) = \alpha_A x + \psi_A y),$$

where β_A (respectively, α_A) is a permutation of the set Q, and φ_A (respectively, ψ_A) is an automorphism of the group $(Q; +)$.

An invertible algebra is called left (right) linear if it is left (right) linear over some group $(Q; +)$.

Definition 4 *An invertible algebra $(Q; \Sigma)$ is called left(right) alinear over a group $(Q; +)$, if every operation $A \in \Sigma$ has the following form:*

$$A(x, y) = \varphi_A x + \beta_A y \quad (A(x, y) = \alpha_A x + \psi_A y),$$

where β_A(respectively, α_A) is a permutation of the set Q, and φ_A(respectively, ψ_A) is an antiautomorphism of the group $(Q; +)$.

An invertible algebra is called left(right) alinear if it is left(right) alinear over some group $(Q; +)$.

Proposition 1 *The invertible algebra $(Q; \Sigma)$ is left linear (alinear) if and only if the dual algebra $(Q; \Sigma^*)$ is right linear (alinear).*

Definition 5 *An invertible algebra $(Q; \Sigma)$ is called linear (alinear) over a group $(Q; +)$ if every operation $A \in \Sigma$ has the following form:*

$$A(x, y) = \varphi_A x + t_A + \psi_A y,$$

where φ_A, ψ_A are automorphisms (antiautomorphisms) of the group $(Q; +)$ and t_A are fixed elements of Q.

An invertible algebra is called linear (alinear) if it is linear (alinear) over some group $(Q; +)$.

Theorem 1 *[9] If the following hyperidentity with permutations*

$$\alpha_1^{A,B}A(\alpha_2^{A,B}B(x,y),z) = B(\alpha_3^{A,B}x, \alpha_4^{A,B}A(\alpha_5^{A,B}y, \alpha_6^{A,B}z)) \tag{1}$$

is valid in the algebra $(Q;\Sigma)$ for all $A,B \in \Sigma$ and for some permutations $\alpha_i^{A,B}(i = 1,\ldots,6)$, then the algebra $(Q;\Sigma)$ is principally isotopic to group.

Conversely, if the invertible algebra $(Q;\Sigma)$ is principally isotopic to group $(Q;\cdot)$, then for all $A,B \in \Sigma$, there exist permutations $\alpha_i^{A,B}(i = 1,\ldots,6)$ such that the hyperidentity with permutations (1) is valid in the algebra $(Q;\Sigma)$.

Theorem 2 *[9] If the following hyperidentity with permutations*

$$\begin{aligned}\alpha_1^{A,B}A(\alpha_2^{A,B}B(\alpha_3^{A,B}x, \alpha_4^{A,B}z), \alpha_5^{A,B}B(\alpha_6^{A,B}w, \alpha_7^{A,B}v)) = \\ A(\alpha_8^{A,B}B(w,z), \alpha_9^{A,B}B(x,v))\end{aligned} \tag{2}$$

is valid in the invertible algebra $(Q;\Sigma)$ for all $A,B \in \Sigma$ and for some permutations $\alpha_i^{A,B}(i = 1,\ldots,9)$, then the algebra $(Q;\Sigma)$ is principally isotopic to an abelian group.

Conversely, if the invertible algebra $(Q;\Sigma)$ is principally isotopic to an abelian group $(Q;\cdot)$, then for all $A,B \in \Sigma$, there exist permutations $\alpha_i^{A,B}(i = 1,\ldots,9)$ such that the hyperidentity with permutations (2) is valid in the algebra $(Q;\Sigma)$.

Theorem 3 *[5] Let the nonempty set Q form a quasigroup under four operations A_i (i=1,2,3,4). If these operations satisfy the following identity:*

$$A_1(A_2(x,y),z) = A_3(x, A_4(y,z)), \tag{3}$$

then there exists an operation $(\cdot)$ under which Q forms a group isotopic to all these four quasigroups.

Theorem 4 *[2] Let the nonempty set Q form a quasigroup under six operations A_i (i=1,2,3,4,5,6). If these operations satisfy the following identity:*

$$A_1(A_2(x,y), A_3(z,u)) = A_4(A_5(x,z), A_6(y,u)), \tag{4}$$

then there exists an operation $(\cdot)$ under which Q forms an abelian group isotopic to all these six quasigroups, i.e.,

$$A_1(x,y) = \alpha x \cdot \beta y, \qquad A_4(x,y) = \chi x \cdot \varphi y,$$

$$A_2(x,y) = \alpha^{-1}(\gamma x \cdot \delta y), \qquad A_5(x,y) = \chi^{-1}(\gamma x \cdot \theta y),$$
$$A_3(x,y) = \beta^{-1}(\theta x \cdot \psi y), \qquad A_6(x,y) = \varphi^{-1}(\delta x \cdot \psi y),$$

where $\alpha, \beta, \gamma, \delta, \chi, \varphi, \psi, \theta$ are permutations of Q.

Theorem 5 *[10] The class of all left linear invertible algebras is characterized by the following second-order formula:*

$$A(B(x,B^{-1}(u,y)),z) = B(A(x,A^{-1}(u,u)),B^{-1}(u,A(y,z))),$$

for all $x,y,u,z \in Q$ and all $A,B \in \Sigma$.

Theorem 6 *[10] The class of all right linear invertible algebras is characterized by the following second-order formula:*

$$B(x,A({}^{-1}A(y,u),z)) = A({}^{-1}A(B(x,y),u),B({}^{-1}B(u,u),z)),$$

for all $x,y,u,z \in Q$ and all $A,B \in \Sigma$.

Theorem 7 *[7] The invertible algebra $(Q;\Sigma)$ is linear if and only if the following second-order formula:*

$$A(B(x,y),B(u,v)) = A(B(x,u),B(\alpha_u^{A,B}y,v)),$$

where

$$\alpha_u^{A,B}y = {}^{-1}B(A^{-1}(u,A(B({}^{-1}B(u,u),y),u)),B^{-1}(u,u))$$

is valid in the algebra $(Q;\Sigma\cup\Sigma^{-1}\cup{}^{-1}\Sigma)$ for all $A,B \in \Sigma$.

Theorem 8 *[7] The invertible algebra $(Q;\Sigma)$ is alinear if and only if the following second-order formula:*

$$A(B(x,y),B(u,v)) = A(B(\beta_x^{A,B}v,y),B(u,x)),$$

where

$$\beta_x^{A,B}v = {}^{-1}B({}^{-1}A(A(x,B({}^{-1}B(x,x),v)),x),B^{-1}(x,x))$$

is valid in the algebra $(Q;\Sigma\cup\Sigma^{-1}\cup{}^{-1}\Sigma)$ for all $A,B \in \Sigma$.

Let $(Q;\Sigma)$ be an invertible algebra and F be a second-order formula. Then F^* is a second-order formula, where all the operations $A \in \Sigma$ are replaced appropriately by the operations $A^* \in \Sigma^*$.

Theorem 9 *If the second-order formula F is valid in the invertible algebra $(Q;\Sigma)$, then the second-order formula F^* is valid in the invertible algebra $(Q;\Sigma^*)$.*

2 Linear invertible algebras

2.1 Left linear invertible algebras

Proposition 2 *If one of the following hyperidentities with permutations*

$$A(B(x,y),z) = B(\alpha_1^{A,B}x, \alpha_2^{A,B}A(\alpha_3^{A,B}y, \alpha_4^{A,B}z)), \tag{5}$$

$$A(B(x,y),z) = A(\alpha_1^{A,B}x, \alpha_2^{A,B}A(\alpha_3^{A,B}y, z)), \tag{6}$$

$$\alpha_1^{A,B}A(B(x,y),z) = B(\alpha_2^{A,B}x, \alpha_3^{A,B}A(\alpha_4^{A,B}y, \alpha_5^{A,B}z)) \tag{7}$$

is valid in the invertible algebra $(Q;\Sigma)$ for all $A, B \in \Sigma$ and for some permutations $\alpha_i^{A,B}(i = 1,\dots,5)$, then the algebra $(Q;\Sigma)$ is left linear.

Proof. First, we prove the statement for identity (5). According to Theorem 1, if the identity (5) is valid in the algebra $(Q;\Sigma)$, then $(Q;\Sigma)$ is principally isotopic to a group, i.e., there exists a group $(Q;+)$ such that for every $A \in \Sigma$

$$A(x,y) = \alpha_A x + \beta_A y,$$

where $\alpha_A, \beta_A \in S_Q$. Let us rewrite the identity (5) by using the last equality:

$$\alpha_A(\alpha_B x + \beta_B y) + \beta_A z = \alpha_B\alpha_1^{A,B}x + \beta_B\alpha_2^{A,B}(\alpha_A\alpha_3^{A,B}y + \beta_A\alpha_4^{A,B}z).$$

By taking $z = \beta_A^{-1}0$ (0 is the neutral element of the group $(Q;+)$) we get:

$$\alpha_A(\alpha_B x + \beta_B y) = \alpha_B\alpha_1^{A,B}x + \beta_B\alpha_2^{A,B}(\alpha_A\alpha_3^{A,B}y + \beta_A\alpha_4^{A,B}\beta_A^{-1}0)$$

or

$$\alpha_A(x + y) = \alpha_B\alpha_1^{A,B}\alpha_B^{-1}x + \beta_B\alpha_2^{A,B}(\alpha_A\alpha_3^{A,B}\beta_B^{-1}y + \beta_A\alpha_4^{A,B}\beta_A^{-1}0).$$

The last equality shows that α_A is quasiautomorphism of the group $(Q;+)$. Then we have

$$\alpha_A x = \varphi_A x + c_A,$$

where $\varphi_A \in Aut(Q;+), c_A \in Q$ and the following equality is valid:

$$A(x,y) = \varphi_A x + c_A + \beta_A y.$$

Since A is an arbitrary operation from Σ, the algebra $(Q;\Sigma)$ is left linear.

Secondly, we will prove the statement of the theorem for the identity (6). Let (6) holds in $(Q;\Sigma)$ for all $A, B \in \Sigma$ and for some permutations $\alpha_i^{A,B}(i = 1,2,3)$. The identity (6) is a particular case of the associativity equation (3), where

$$A_1(x,y) = A(x,y), \qquad A_2(x,y) = B(x,y),$$

$$A_3(x,y) = A(\alpha_1^{A,B}x, y), \qquad A_4(x,y) = \alpha_2^{A,B}A(\alpha_3^{A,B}x, y).$$

According to Theorem 3, the quasigroups A_1, A_2, A_3, A_4 are isotopic to the same group $(Q;\cdot)$, hence, the operations A, B are isotopic to $(Q;\cdot)$. Since the operations A and B are arbitrary, all operations from Σ are isotopic to this group.

For every $X \in \Sigma$, let us define the operation:

$$x \underset{X}{+} y = X(R_{X,a}^{-1}x, L_{X,b}^{-1}y), \tag{8}$$

where a, b are some fixed elements from Q. The operation $\underset{X}{+}$ is a loop operation with the identity element $0_X = X(b,a)$. Obviously, $(Q;\underset{X}{+})$ is a loop isotopic to the group $(Q;\cdot)$. Hence, by Albert's theorem, it is a group. For every $X \in \Sigma$, each $(Q;\underset{X}{+})$ is a group. Hence, (6) can be rewritten in the following form:

$$R_{A,a}(R_{B,a}x \underset{B}{+} L_{B,b}y) \underset{A}{+} L_{A,b}z = R_{A,a}\alpha_1^{A,B}x \underset{A}{+} L_{A,b}\alpha_2^{A,B}(R_{A,a}\alpha_3^{A,B}y \underset{A}{+} L_{A,b}z).$$

Taking $z = L_{A,b}^{-1}0_A$ in the last equality, we have

$$R_{A,a}(R_{B,a}x \underset{B}{+} L_{B,b}y) = R_{A,a}\alpha_1^{A,B}x \underset{A}{+} L_{A,b}\alpha_2^{A,B}R_{A,a}\alpha_3^{A,B}y,$$

$$R_{A,a}(x \underset{B}{+} y) = \alpha_{A,B}x \underset{A}{+} \beta_{A,B}y, \tag{9}$$

where $\alpha_{A,B} = R_{A,a}\alpha_1^{A,B}R_{B,a}^{-1}$, $\beta_{A,B} = L_{A,b}\alpha_2^{A,B}R_{A,a}\alpha_3^{A,B}L_{B,b}^{-1}$ are permutations of the set Q. Since the operations A and B are arbitrary, we can take $A = B$ in (9). Hence,

$$R_{A,a}(x \underset{A}{+} y) = \alpha_{A,A}x \underset{A}{+} \beta_{A,A}y. \tag{10}$$

From (9) and (10) we have

$$\alpha_{A,B}^{-1}x \underset{B}{+} \beta_{A,B}^{-1}y = \alpha_{A,A}^{-1}x \underset{A}{+} \beta_{A,A}^{-1}y,$$

$$x \underset{A}{+} y = \gamma_{A,B}x \underset{B}{+} \delta_{A,B}y, \tag{11}$$

where $\gamma_{A,B} = \alpha_{A,B}^{-1}\alpha_{A,A}$ and $\delta_{A,B} = \beta_{A,B}^{-1}\beta_{A,A}$ are permutations of the set Q. Hence, together with (11) we get

$$R_{A,a}(x \underset{B}{+} y) = \gamma_{A,B}\alpha_{A,B}x \underset{B}{+} \delta_{A,B}\beta_{A,B}y,$$

i.e., $R_{A,a}$ is a quasiautomorphism of the group $(Q;\underset{B}{+})$. Since A is arbitrary, $R_{A,a}$ is a quasiautomorphism of the group $(Q;\underset{B}{+})$ for all operations $A\in\Sigma$.

According to (8) we have

$$A(x,y)=R_{A,a}x\underset{A}{+}L_{A,b}y.$$

This, according to (11), gives:

$$A(x,y)=\theta^1_{A,B}x\underset{B}{+}\theta^2_{A,B}y, \tag{12}$$

where $\theta^1_{A,B}=\gamma_{A,B}R_{A,a}$ and $\theta^2_{A,B}=\delta_{A,B}L_{A,b}$ are permutations of the set Q. Thus, we can represent every operation from Σ by the operation $\underset{B}{+}$.

Let $+=\underset{B}{+}$. We prove that $\theta^1_{A,B}$ is a quasiautomorphism of the group $(Q;+)$. To do it, take $z=(\theta^2_{A,B})^{-1}0_B$ in (8) and rewrite this equality in the form:

$$\theta^1_{A,B}(R_{B,a}x+L_{B,b}y)+\theta^2_{A,B}z=\theta^1_{A,B}\alpha_1^{A,B}x+\theta^2_{A,B}\alpha_2^{A,B}(\theta^1_{A,B}\alpha_3^{A,B}y+\theta^2_{A,B}z),$$

$$\theta^1_{A,B}(R_{B,a}x+L_{B,b}y)=\theta^1_{A,B}\alpha_1^{A,B}x+\theta^2_{A,B}\alpha_2^{A,B}\theta^1_{A,B}\alpha_3^{A,B}y.$$

The last equality shows that $\theta^1_{A,B}$ is a quasiautomorphism of the group $(Q;+)$. According to [[3], Lemma 2.5] we have

$$\theta^1_{A,B}x=\varphi_Ax+s_A,$$

where φ_A is an automorphism of the group $(Q;+)$ and s_A is some element of the set Q. Hence, it follows from (12) that

$$A(x,y)=\varphi_Ax+\beta_Ay, \tag{13}$$

where $\beta_Ay=s_A+\theta^2_{A,B}y$. Since A is an arbitrary operation, we obtain that all operations from Σ can be represented in the from (13), i.e., the algebra $(Q;\Sigma)$ is left linear.

Finally, here is the proof for the identity (7). According to Theorem 4, if the identity (7) is valid in the algebra $(Q;\Sigma)$, then $(Q;\Sigma)$ is principally isotopic to a group, i.e., there exists a group $(Q;+)$ such that for every $A\in\Sigma$

$$A(x,y)=\alpha_Ax+\beta_Ay,$$

where $\alpha_A,\beta_A\in S_Q$. Let us rewrite the identity (7) by using the last equality:

$$\alpha_1^{A,B}(\alpha_A(\alpha_Bx+\beta_By)+\beta_Az)=\alpha_B\alpha_2^{A,B}x+\beta_B\alpha_3^{A,B}(\alpha_A\alpha_4^{A,B}y+\beta_A\alpha_5^{A,B}z).$$

By doing the following replacements $x=\alpha_B^{-1}x, y=\beta_B^{-1}y, z=\beta_A^{-1}z$ we get:

$$\alpha_1^{A,B}(\alpha_A(x+y)+z)=\alpha_B\alpha_2^{A,B}\alpha_B^{-1}x+\beta_B\alpha_3^{A,B}(\alpha_A\alpha_4^{A,B}\beta_B^{-1}y+\beta_A\alpha_5^{A,B}\beta_A^{-1}z).$$

According to Lemma 1, α_A is quasiautomorphism of the group $(Q;+)$, i.e.,

$$\alpha_A x = \varphi_A x + c_A,$$

where $\varphi_A \in Aut(Q;+), c_A \in Q$. Then the following equality is valid:

$$A(x,y) = \varphi_A x + c_A + \beta_A y.$$

Since A is an arbitrary operation from Σ, the algebra $(Q;\Sigma)$ is left linear. □

Theorem 10 *The invertible algebra $(Q;\Sigma)$ is left linear if and only if for all $A,B \in \Sigma$ and for some permutations $\alpha_i^{A,B} (i = 1,\dots,4)$ the hyperidentity with permutations (5) is valid in the algebra $(Q;\Sigma)$.*

Proof. If the invertible algebra $(Q;\Sigma)$ is left linear, then according to Theorem 5 the following formula is valid in that algebra:

$$A(B(x,B^{-1}(u,y)),z) = B(A(x,A^{-1}(u,u)),B^{-1}(u,A(y,z))).$$

The formula may be rewritten in the following form:

$$A(B(x,y),z) = B(R_{A,A^{-1}(u,u)}x, L_{B,u}^{-1}A(L_{B,u}y,z)).$$

By taking $u = a$, where a is a fixed element from Q we get the identity (5), where

$$\alpha_1^{A,B} = R_{A,A^{-1}(a,a)},\ \alpha_2^{A,B} = L_{B,a}^{-1},\ \alpha_3^{A,B} = L_{B,a},\ \alpha_4^{A,B} = \epsilon.$$

The sufficiency follows from the Proposition 2. □

2.2 Right linear invertible algebras

Proposition 3 *If one of the following hyperidentities with permutations*

$$A(x,B(y,z)) = A(\alpha_1^{A,B}A(x,\alpha_2^{A,B}y),\alpha_3^{A,B}z), \tag{14}$$

$$A(x,B(y,z)) = A(\alpha_1^{A,B}A(\alpha_2^{A,B}x,\alpha_3^{A,B}y),\alpha_4^{A,B}z), \tag{15}$$

$$\alpha_1^{A,B}A(x,B(y,z)) = B(\alpha_2^{A,B}A(\alpha_3^{A,B}x,\alpha_4^{A,B}y),\alpha_5^{A,B}z) \tag{16}$$

is valid in the invertible algebra $(Q;\Sigma)$ for all $A,B \in \Sigma$ and for some permutations $\alpha_i^{A,B} (i = 1,\dots,4)$, then the algebra $(Q;\Sigma)$ is right linear.

Proof. Follows from the Proposition 1 and the Theorem 9. □

Theorem 11 *The invertible algebra $(Q;\Sigma)$ is right linear if and only if for all $A,B \in \Sigma$ and for some permutations $\alpha_i^{A,B} (i = 1,\dots,4)$ the hyperidentity with permutations (15) is valid in the algebra $(Q;\Sigma)$.*

Proof. If the invertible algebra $(Q;\Sigma)$ is right linear, then according to Theorem 6 the following formula is valid in that algebra:

$$B(x, A(^{-1}A(y,u),z)) = A(^{-1}A(B(x,y),u), B(^{-1}B(u,u),z)).$$

We may rewrite the formula in the following form:

$$B(x, A(y,z)) = A(R^{-1}_{A,u}B(x, R_{A,u}y), R_{B,B^{-1}(u,u)}z).$$

By taking $u = a$, where a is a fixed element from Q, we get the identity (15), where

$$\alpha_1^{A,B} = R^{-1}_{A,a}, \alpha_2^{A,B} = \epsilon, \alpha_3^{A,B} = R_{A,a}, \alpha_4^{A,B} = R_{B,B^{-1}(a,a)}.$$

The sufficiency follows from the Proposition 3. □

2.3 Linear invertible algebras

Proposition 4 *If one of the following hyperidentities with permutations*

$$\alpha_1^{A,B}A(B(x,y),z) = B(\alpha_2^{A,B}x, A(\alpha_3^{A,B}y, \alpha_4^{A,B}z)), \tag{17}$$
$$\alpha_1^{A,B}A(B(x,y),z) = B(\alpha_2^{A,B}x, \alpha_3^{A,B}A(y, \alpha_4^{A,B}z)), \tag{18}$$
$$A(\alpha_1^{A,B}B(x,y),z) = B(\alpha_2^{A,B}x, A(y, \alpha_3^{A,B}z)) \tag{19}$$

is valid in the invertible algebra $(Q;\Sigma)$ for all $A, B \in \Sigma$ and for some permutations $\alpha_i^{A,B}(i = 1,\ldots,4)$, then the algebra $(Q;\Sigma)$ is linear.

Proof. According to Theorem 1, if each of the above-mentioned identities is valid in the algebra $(Q;\Sigma)$, then $(Q;\Sigma)$ is principally isotopic to a group, i.e., there exists a group $(Q;+)$ such that for every $A \in \Sigma$

$$A(x,y) = \alpha_A x + \beta_A y, \tag{20}$$

where $\alpha_A, \beta_A \in S_Q$.

At first, we will prove the proposition for the identity (17). Let us rewrite the identity (17) by using (20):

$$\alpha_1^{A,B}(\alpha_A(\alpha_B x + \beta_B y) + \beta_A z) = \alpha_B\alpha_2^{A,B}x + \beta_B(\alpha_A\alpha_3^{A,B}y + \beta_A\alpha_4^{A,B}z).$$

After replacement of variables we get:

$$\alpha_1^{A,B}(\alpha_A(x+y) + z) = \alpha_B\alpha_2^{A,B}\alpha_B^{-1}x + \beta_B(\alpha_A\alpha_3^{A,B}\beta_B^{-1}y + \beta_A\alpha_4^{A,B}\beta_A^{-1}z).$$

According to Lemma 1, α_A and β_B are quasiautomorphisms of the group $(Q;+)$ for every $A, B \in \Sigma$, i.e.,

$$\alpha_A x = \varphi_A x + c_A, \quad \beta_A y = d_A + \psi_A x$$

where $\varphi_A, \psi_A \in Aut(Q;+), c_A, d_A \in Q$. Thus,

$$A(x,y) = \varphi_A x + u_A + \psi_A y,$$

where $u_A = c_A + d_A$. Hence, the algebra $(Q;\Sigma)$ is linear.

Secondly, we will prove the proposition for the identity (18). Let us rewrite the identity (18) by using (20):

$$\alpha_1^{A,B}(\alpha_A(\alpha_B x + \beta_B y) + \beta_A z) = \alpha_B \alpha_2^{A,B} x + \beta_B \alpha_3^{A,B}(\alpha_A y + \beta_A \alpha_4^{A,B} z).$$

After replacement of variables we get:

$$\alpha_1^{A,B}(\alpha_A(x+y) + z) = \alpha_B \alpha_2^{A,B} \alpha_B^{-1} x + \beta_B \alpha_3^{A,B}(\alpha_A \beta_B^{-1} y + \beta_A \alpha_4^{A,B} \beta_A^{-1} z).$$

According to Lemma 1, α_A and $\alpha_A \beta_B^{-1}$ are quasiautomorphisms of the group $(Q;+)$ for every $A, B \in \Sigma$, thus β_B is a quasiautomorphism as well (the set of quasiautomorphisms forms a group). Hence, the algebra $(Q;\Sigma)$ is linear.

Finally, here is the proof of the proposition for the identity (19). Let us rewrite the identity (19) by using (20):

$$\alpha_A \alpha_1^{A,B}(\alpha_B x + \beta_B y) + \beta_A z = \alpha_B \alpha_2^{A,B} x + \beta_B(\alpha_A y + \beta_A \alpha_3^{A,B} z).$$

After replacement of variables we get:

$$\alpha_A \alpha_1^{A,B}(x+y) + z = \alpha_B \alpha_2^{A,B} \alpha_B^{-1} x + \beta_B(\alpha_A \beta_B^{-1} y + \beta_A \alpha_3^{A,B} \beta_A^{-1} z).$$

According to Lemma 1, β_B and $\alpha_A \beta_B^{-1}$ are quasiautomorphisms of the group $(Q;+)$ for every $A, B \in \Sigma$, thus, α_A is a quasiautomorphism as well (the set of quasiautomorphisms forms a group). Hence, the algebra $(Q;\Sigma)$ is linear. □

Theorem 12 *The invertible algebra $(Q;\Sigma)$ is linear if and only if for all $A, B \in \Sigma$ and for some permutations $\alpha_i^{A,B} (i = 1, \ldots, 3)$ the following hyperidentity with permutations*

$$A(B(x,y),z) = A(\alpha_1^{A,B} x, B(\alpha_2^{A,B} y, \alpha_3^{A,B} z)), \tag{21}$$

is valid in the algebra $(Q;\Sigma)$.

Proof. Let (21) holds in $(Q;\Sigma)$ for all $A, B \in \Sigma$ and for some permutations $\alpha_i^{A,B} (i = 1, 2, 3)$. The identity (21) is a particular case of the associativity equation (3), where

$$A_1(x,y) = A(x,y), \qquad A_2(x,y) = B(x,y),$$

$$A_3(x,y) = A(\alpha_1^{A,B} x, y), \qquad A_4(x,y) = B(\alpha_2^{A,B} x, \alpha_3^{A,B} y).$$

According to Theorem 3, the quasigroups A_1, A_2, A_3, A_4 are isotopic to the same group $(Q;\cdot)$, hence, the operations A, B are isotopic to $(Q;\cdot)$. Since the operations A and B are arbitrary, we obtain that all operations from Σ are isotopic to this group.

For every $X \in \Sigma$, let us define the operation:

$$x \underset{X}{+} y = X(R^{-1}_{X,a}x, L^{-1}_{X,b}y), \tag{22}$$

where a, b are some fixed elements from Q. The operation $\underset{X}{+}$ is a loop operation with the identity element $0_X = X(b, a)$. Obviously, $(Q; \underset{X}{+})$ is a loop isotopic to the group $(Q;\cdot)$, Hence, by Albert's theorem, it is a group. For every $X \in \Sigma$, each $(Q; \underset{X}{+})$ is a group. Thus, (21) can be rewritten in the form:

$$R_{A,a}(R_{B,a}x \underset{B}{+} L_{B,b}y) \underset{A}{+} L_{A,b}z = R_{A,a}\alpha_1^{A,B}x \underset{A}{+} L_{A,b}(R_{A,a}\alpha_2^{A,B}y \underset{A}{+} L_{A,b}\alpha_3^{A,B}z).$$

Taking $z = L^{-1}_{A,b}0_A$ in the last equality, we have

$$R_{A,a}(R_{B,a}x \underset{B}{+} L_{B,b}y) = R_{A,a}\alpha_1^{A,B}x \underset{A}{+} L_{A,b}(R_{A,b}\alpha_2^{A,B}y \underset{A}{+} L_{A,b}\alpha_3^{A,B}L^{-1}_{A,b}0).$$

After replacement of parameters we get

$$R_{A,a}(x \underset{B}{+} y) = \alpha_{A,B}x \underset{A}{+} \beta_{A,B}y, \tag{23}$$

where $\alpha_{A,B}$ and $\beta_{A,B}$ are permutations of the set Q. Since the operations A and B are arbitrary, we can take $A = B$ in (23). Hence,

$$R_{A,a}(x \underset{A}{+} y) = \alpha_{A,A}x \underset{A}{+} \beta_{A,A}y. \tag{24}$$

From (23) and (24) we have

$$\alpha^{-1}_{A,B}x \underset{B}{+} \beta^{-1}_{A,B}y = \alpha^{-1}_{A,A}x \underset{A}{+} \beta^{-1}_{A,A}y,$$

$$x \underset{A}{+} y = \gamma_{A,B}x \underset{B}{+} \delta_{A,B}y, \tag{25}$$

where $\gamma_{A,B} = \alpha^{-1}_{A,B}\alpha_{A,A}$ and $\delta_{A,B} = \beta^{-1}_{A,B}\beta_{A,A}$ are permutations of the set Q. Hence, according to (25) we get

$$R_{A,a}(x \underset{B}{+} y) = \gamma_{A,B}\alpha_{A,B}x \underset{B}{+} \delta_{A,B}\beta_{A,B}y,$$

i.e., $R_{A,a}$ is a quasiautomorphism of the group $(Q; \underset{B}{+})$. Since A is arbitrary, $R_{A,a}$ is a quasiautomorphism of the group $(Q; \underset{B}{+})$ for all operations $A \in \Sigma$.

According to (22) we have

$$A(x,y) = R_{A,a}x \underset{A}{+} L_{A,b}y.$$

This, according to (25), gives:

$$A(x,y) = \theta^1_{A,B}x \underset{B}{+} \theta^2_{A,B}y, \tag{26}$$

where $\theta^1_{A,B} = \gamma_{A,B}R_{A,a}$ and $\theta^2_{A,B} = \delta_{A,B}L_{A,b}$ are permutations of the set Q. Thus, we can represent every operation from Σ by the operation $\underset{B}{+}$. Let us fix B and denote $+ = \underset{B}{+}$, then we can write (26) in the following way:

$$A(x,y) = \theta^1_A x + \theta^2_A y, \tag{27}$$

Let us rewrite (21) by using (27):

$$\theta^1_A(\theta^1_B x + \theta^2_B y) + \theta^2_A z = \theta^1_A \alpha^{A,B}_1 x + \theta^2_A(\theta^1_A \alpha^{A,B}_2 y + \theta^2_A \alpha^{A,B}_3 z)$$

According to Lemma 1, θ^1_A and θ^2_A are quasiautomorphisms of the group $(Q;+)$ and thus

$$A(x,y) = \varphi_A x + c_A + \psi_A y,$$

where $\varphi_A, \psi_A \in Aut(Q;+), c_A \in Q$.

Conversely, if the invertible algebra $(Q;\Sigma)$ is linear, then according to Theorem 7 the following second-order formula is valid in that algebra:

$$A(B(x,y),B(u,z)) = A(B(x,u),B(\alpha^{A,B}_u y,z)),$$

where

$$\alpha^{A,B}_u y = {}^{-1}B(A^{-1}(u, A(B({}^{-1}B(u,u),y),u)), B^{-1}(u,u)).$$

The formula may be rewritten in the following form:

$$A(B(x,y), L_{B,u}z) = A(R_{B,u}x, B(\alpha^{A,B}_u y, z)).$$

By taking $u = a$, where a is a fixed element from Q, we get the identity (21), where

$$\alpha^{A,B}_1 = R_{B,a}, \alpha^{A,B}_2 = \alpha^{A,B}_a, \alpha^{A,B}_3 = L^{-1}_{B,a}.$$

□

3 Alinear invertible algebras

3.1 Left alinear invertible algebras

Proposition 5 *If the following hyperidentity with permutations*

$$A(B(x,y),z) = B(\alpha_1^{A,B}y, \alpha_2^{A,B}A(\alpha_3^{A,B}z, \alpha_4^{A,B}x)), \tag{28}$$

is valid in the invertible algebra $(Q;\Sigma)$ *for all* $A, B \in \Sigma$ *and for some permutations* $\alpha_i^{A,B}(i = 1,\dots,4)$*, then the algebra* $(Q;\Sigma)$ *is left alinear.*

Proof. Let us rewrite the identity (28) in the following way:

$$A^*(z, B(x,y)) = B^*(\alpha_2^{A,B}A(\alpha_3^{A,B}z, \alpha_4^{A,B}x), \alpha_1^{A,B}y),$$

where $X^*(x,y) = X(y,x)$. According to Theorem 1, $\forall A \in \Sigma$ is principally isotopic to the group $(Q;+)$,i.e. for every $A \in \Sigma$:

$$A(x,y) = \alpha_A x + \beta_A y,$$

where $\alpha_A, \beta_A \in S_Q$. Let's rewrite the identity (28) by using the last equality:

$$\alpha_A(x+y) + z = \alpha_B\alpha_1^{A,B}\beta_B^{-1}y + \beta_B\alpha_2^{A,B}(\alpha_A\alpha_3^{A,B}\beta_A^{-1}z + \beta_A\alpha_4^{A,B}\alpha_B^{-1}x).$$

By taking $z = 0$ (0 is the neutral element of the group $(Q;+)$) we get:

$$\alpha_A(x+y) = \alpha_B\alpha_1^{A,B}\beta_B^{-1}y + \beta_B\alpha_2^{A,B}(\alpha_A\alpha_3^{A,B}\beta_A^{-1}0 + \beta_A\alpha_4^{A,B}\alpha_B^{-1}x).$$

The last equality shows that α_A is an antiquasiautomorphism of the group $(Q;+)$ for every $A, B \in \Sigma$, i.e.

$$\alpha_A x = \overline{\varphi_A}x + c_A,$$

where $\overline{\varphi_A}$ is an antiautomorphism of $(Q;+)$ and $c_A \in Q$. Thus,

$$A(x,y) = \overline{\varphi_A}x + c_A + \beta_A y.$$

Since A is an arbitrary operation from Σ, we obtain that the algebra $(Q;\Sigma)$ is left alinear. $\square$

3.2 Right alinear invertible algebras

Proposition 6 *If the following hyperidentity with permutations*

$$A(\alpha_1^{A,B}B(x,y),z) = B(\alpha_2^{A,B}y, A(\alpha_3^{A,B}z, \alpha_4^{A,B}x)), \tag{29}$$

is valid in the invertible algebra $(Q;\Sigma)$ *for all* $A, B \in \Sigma$ *and for some permutations* $\alpha_i^{A,B}(i = 1,\dots,4)$*, then the algebra* $(Q;\Sigma)$ *is right alinear.*

Proof. Follows from the Proposition 1 and the Theorem 9. $\square$

3.3 Alinear invertible algebras

Proposition 7 *If the following hyperidentity with permutations*

$$A(B(x,y),z) = B(\alpha_1^{A,B}y, A(\alpha_2^{A,B}z, \alpha_3^{A,B}x)) \tag{30}$$

is valid in the invertible algebra $(Q;\Sigma)$ for all $A, B \in \Sigma$ and for some permutations $\alpha_i^{A,B}(i = 1,\ldots,3)$, then the algebra $(Q;\Sigma)$ is alinear.

Proof. Let us rewrite the identity (30) in the following way:

$$A^*(z, B(x,y)) = B^*(A(\alpha_2^{A,B}z, \alpha_3^{A,B}x), \alpha_1^{A,B}y),$$

where $X^*(x,y) = X(y,x)$. According to Theorem 1, every $A \in \Sigma$ is principally isotopic to the group $(Q;+)$, i.e., for every $A \in \Sigma$:

$$A(x,y) = \alpha_A x + \beta_A y,$$

where $\alpha_A, \beta_A \in S_Q$. Let us rewrite the identity (30) by using the last equality:

$$\alpha_A(\alpha_B x + \beta_B y) + \beta_B z = \alpha_B\alpha_1^{A,B}y + \beta_B(\alpha_A\alpha_2^{A,B}z + \beta_A\alpha_3^{A,B}x). \tag{31}$$

By taking $z = \beta_B^{-1}0$ in (31) (0 is the neutral element of the group $(Q;+)$) we get:

$$\alpha_A(\alpha_B x + \beta_B y) = \alpha_B\alpha_1^{A,B}y + \beta_B(\alpha_A\alpha_2^{A,B}\beta_B^{-1}0 + \beta_A\alpha_3^{A,B}x).$$

The last equality shows that α_A is an antiquasiautomorphism of the group $(Q;+)$ for every $A, B \in \Sigma$. Further, let us take $y = (\alpha_1^{A,B})^{-1}\alpha_B^{-1}0$ in (31):

$$(\alpha_A(\alpha_B x + \beta_B(\alpha_1^{A,B})^{-1}\alpha_B^{-1}0) + \beta_B z = \beta_B(\alpha_A\alpha_2^{A,B}z + \beta_A\alpha_3^{A,B}x).$$

The last equality shows that β_B is an antiquasiautomorphism of the group $(Q;+)$ for every $A, B \in \Sigma$, too. Thus,

$$A(x,y) = \overline{\varphi_A}x + c_A + \overline{\psi_A}y,$$

where $\overline{\psi_A}, \overline{\varphi_A}$ are antiautomorphisms of $(Q;+)$ and $c_A \in Q$.

Since A is an arbitrary operation from Σ, the algebra $(Q;\Sigma)$ is alinear.
□

Theorem 13 *The invertible algebra $(Q;\Sigma)$ is alinear if and only if for all $A, B \in \Sigma$ and for some permutations $\alpha_i^{A,B}(i = 1,\ldots,3)$ the following hyperidentity with permutations*

$$A(B(x,y),z) = A(\alpha_1^{A,B}y, B(\alpha_2^{A,B}z, \alpha_3^{A,B}x)), \tag{32}$$

is valid in the algebra $(Q;\Sigma)$.

Proof. Let us rewrite the identity (32) in the following way:

$$A^*(z, B(x,y)) = A^*(B(\alpha_2^{A,B}z, \alpha_3^{A,B}x), \alpha_1^{A,B}y).$$

From the last equality we can conclude that $\forall A \in \Sigma$ is isotopic to a group. Similarly to the case of linear invertible algebras, we get that every operation

$$x \underset{A}{+} y = A(R_{A,a}^{-1}x, L_{A,b}^{-1}y) \tag{33}$$

is a group with the identity element $0_X = X(b,a)$. Let us rewrite the identity (32) by using equality (33):

$$\begin{gathered} R_{A,a}(R_{B,a}x \underset{B}{+} L_{B,b}y) \underset{A}{+} L_{A,b}z = \\ R_{A,a}\alpha_1^{A,B}y \underset{A}{+} L_{A,b}(R_{A,a}\alpha_2^{A,B}z \underset{A}{+} L_{A,b}\alpha_3^{A,B}x). \end{gathered} \tag{34}$$

By taking $z = L_{A,b}^{-1}0_A$ in (34) we get:

$$R_{A,a}(R_{B,a}x \underset{B}{+} L_{B,b}y) = R_{A,a}\alpha_1^{A,B}y \underset{A}{+} L_{A,b}(R_{A,a}\alpha_2^{A,B}L_{A,b}^{-1}0_A \underset{A}{+} L_{A,b}\alpha_3^{A,B}x)$$

or

$$R_{A,a}(x \underset{B}{+} y) = \alpha_{A,B}y \underset{A}{+} \beta_{A,B}x, \tag{35}$$

where $\alpha_{A,B}, \beta_{A,B} \in S_Q$. Since the operations A and B are arbitrary, we can take $A = B$ in (35). Hence,

$$R_{A,a}(x \underset{A}{+} y) = \alpha_{A,A}y \underset{A}{+} \beta_{A,A}x. \tag{36}$$

From (35) and (36) we have

$$\alpha_{A,B}^{-1}x \underset{B}{+} \beta_{A,B}^{-1}y = \alpha_{A,A}^{-1}x \underset{A}{+} \beta_{A,A}^{-1}y,$$

$$x \underset{A}{+} y = \gamma_{A,B}x \underset{B}{+} \delta_{A,B}y, \tag{37}$$

where $\gamma_{A,B}, \delta_{A,B} \in S_Q$. Hence, according to (37) we get

$$R_{A,a}(x \underset{B}{+} y) = \gamma_{A,B}\alpha_{A,B}x \underset{B}{+} \delta_{A,B}\beta_{A,B}y,$$

i.e., $R_{A,a}$ is a quasiautomorphism of the group $(Q; \underset{B}{+})$. Since A is arbitrary, $R_{A,a}$ is a quasiautomorphism of the group $(Q; \underset{B}{+})$ for all operations $A \in \Sigma$.

According to (33) we have

$$A(x,y) = R_{A,a}x \underset{A}{+} L_{A,b}y.$$

This, according to (37), gives:

$$A(x,y) = \theta^1_{A,B}x \underset{B}{+} \theta^2_{A,B}y, \tag{38}$$

where $\theta^1_{A,B} = \gamma_{A,B}R_{A,a}$ and $\theta^2_{A,B} = \delta_{A,B}L_{A,b}$ are permutations of the set Q. Thus, we can represent every operation from Σ by the operation $\underset{B}{+}$.Let us fix B and denote $+ = \underset{B}{+}$, then we can write (38) in the following way:

$$A(x,y) = \theta^1_A x + \theta^2_A y. \tag{39}$$

Let us rewrite the identity (32) by using (39):

$$\theta^1_A(\theta^1_B x + \theta^2_B y) + \theta^2_A z = \theta^1_A \alpha_1^{A,B} y + \theta^2_A(\theta^1_B \alpha_2^{A,B} z + \theta^2_B \alpha_3^{A,B} x). \tag{40}$$

By taking $z = (\theta^2_A)^{-1}0$ in (38) (0 is the neutral element of the group $(Q;+)$) we get:

$$\theta^1_A(\theta^1_B x + \theta^2_B y) = \theta^1_A \alpha_1^{A,B} y + \theta^2_A(\theta^1_B \alpha_2^{A,B} (\theta^2_A)^{-1}0 + \theta^2_B \alpha_3^{A,B} x).$$

The last equality shows that θ^1_A is an antiquasiautomorphism of the group $(Q;+)$ for every $A, B \in \Sigma$. Further, let us take $y = (\alpha_1^{A,B})^{-1}(\theta^2_A)^{-1}0$ in (38):

$$\theta^1_A(\theta^1_B x + \theta^2_B (\theta^2_A)^{-1}0) + \theta^2_A z = \theta^2_A(\theta^1_B \alpha_2^{A,B} z + \theta^2_B \alpha_3^{A,B} x).$$

The last equality shows that θ^2_A is an antiquasiautomorphism of the group $(Q;+)$ for every $A, B \in \Sigma$, too. Thus

$$A(x,y) = \overline{\varphi_A}x + c_A + \overline{\psi_A}y,$$

where $\overline{\psi_A}, \overline{\varphi_A}$ are antiautomorphisms of $(Q;+)$ and $c_A \in Q$. Since A is an arbitrary operation from Σ, the algebra $(Q;\Sigma)$ is alinear.

Conversely, if the invertible algebra $(Q;\Sigma)$ is alinear, then according to Theorem 8 the following second-order formula is valid in that algebra:

$$A(B(x,y), B(u,v)) = A(B(\beta_x^{A,B}v, y), B(u,x)),$$

where

$$\beta_x^{A,B}v = {}^{-1}B({}^{-1}A(A(x, B({}^{-1}B(x,x), v)), x), B^{-1}(x,x)),$$

The formula may be rewritten in the following form:

$$A(L_{B,x}y, B(u,v)) = A(B(\beta_x^{A,B}v, y), R_{B,x}u).$$

By taking $x = a$, where a is a fixed element from Q we get the identity (32), where

$$\alpha_1^{A,B} = L_{B,a}, \alpha_2^{A,B} = R^{-1}_{B,a}, \alpha_3^{A,B} = (\beta_a)^{-1}.$$

□

4 Invertible algebras of mixed type of linearity

Definition 6 *An invertible algebra $(Q;\Sigma)$ is called an invertible algebra of mixed type of linearity of the first (second) kind over a group $(Q;+)$, if every operation $A\in\Sigma$ has the form*

$$A(x,y)=\varphi_A x+c_A+\overline{\psi}_A y \quad (A(x,y)=\overline{\varphi}_A x+c_A+\psi_A y),$$

where $\varphi_A,\psi_A\in Aut(Q;+),\overline{\psi}_A,\overline{\varphi}_A$ are antiautomorphisms of $(Q;+)$, and c_A is a fixed element from Q.

4.1 Invertible algebras of mixed type of linearity of the first kind

Proposition 8 *If the following hyperidentity with permutations*

$$A(B(x,y),z)=B(\alpha_1^{A,B}x,A(\alpha_2^{A,B}z,\alpha_3^{A,B}y)), \tag{41}$$

is valid in the invertible algebra $(Q;\Sigma)$ for all $A,B\in\Sigma$ and for some permutations $\alpha_i^{A,B}(i=1,\ldots,3)$, then the algebra $(Q;\Sigma)$ is mixed type of linearity of the first kind.

Proof. Let us rewrite the identity (41) in the following way:

$$A(B(x,y),z)=B(\alpha_1^{A,B}x,A^*(\alpha_3^{A,B}y,\alpha_2^{A,B}z)),$$

where $X^*(x,y)=X(y,x)$. According to Theorem 1, every $A\in\Sigma$ is principally isotopic to the group $(Q;+)$, i.e., for every $A\in\Sigma$:

$$A(x,y)=\alpha_A x+\beta_A y,$$

where $\alpha_A,\beta_A\in S_Q$. Let us rewrite the identity (41) by using the last equality:

$$\alpha_A(\alpha_B x+\beta_B y)+\beta_B z=\alpha_B\alpha_1^{A,B}x+\beta_B(\alpha_A\alpha_2^{A,B}z+\beta_A\alpha_3^{A,B}y). \tag{42}$$

By taking $x=(\alpha_1^{A,B})^{-1}\alpha_B^{-1}0$ in (42) (0 is the neutral element of the group $(Q;+)$) we get:

$$\alpha_A(\alpha_B(\alpha_1^{A,B})^{-1}\alpha_B^{-1}0+\beta_B y)+\beta_B z=\beta_B(\alpha_A\alpha_2^{A,B}z+\beta_A\alpha_3^{A,B}y).$$

The last equality shows that β_B is an antiquasiautomorphism of the group $(Q;+)$ for every $A,B\in\Sigma$. Further, let us take $z=\beta_B^{-1}0$ in (42):

$$\alpha_A(\alpha_B x+\beta_B y)=\alpha_B\alpha_1^{A,B}x+\beta_B(\alpha_A\alpha_2^{A,B}\beta_B^{-1}0+\beta_A\alpha_3^{A,B}y).$$

The last equality shows that α_A is a quasiautomorphism of the group $(Q;+)$ for every $A, B \in \Sigma$. Thus,

$$A(x,y) = \varphi_A x + c_A + \overline{\psi_A} y,$$

where $\overline{\psi_A}$ is an antiautomorphism of $(Q;+)$ and $\varphi_A \in Aut(Q;+), c_A \in Q$. Since A is an arbitrary operation from Σ, the algebra $(Q;\Sigma)$ is a mixed type of linearity of the first kind. □

4.2 Invertible algebras of mixed type of linearity of the second kind

Proposition 9 *If the following hyperidentity with permutations*

$$A(B(x,y),z) = B(\alpha_1^{A,B} y, A(\alpha_2^{A,B} x, \alpha_3^{A,B} z)), \tag{43}$$

is valid in the invertible algebra $(Q;\Sigma)$ for all $A, B \in \Sigma$ and for some permutations $\alpha_i^{A,B} (i = 1, \ldots, 3)$, then the algebra $(Q;\Sigma)$ is mixed type of linearity of the second kind.

Proof. Let us rewrite the identity (43) in the following way:

$$A(B^*(x,y),z) = B(\alpha_1^{A,B} y, A(\alpha_2^{A,B} x, \alpha_3^{A,B} z)),$$

where $X^*(x,y) = X(y,x)$. According to Theorem 1, every $A \in \Sigma$ is principally isotopic to the group $(Q;+)$, i.e., for every $A \in \Sigma$:

$$A(x,y) = \alpha_A x + \beta_A y,$$

where $\alpha_A, \beta_A \in S_Q$. Let us rewrite the identity (43) by using the last equality:

$$\alpha_A(\alpha_B x + \beta_B y) + \beta_A z = \alpha_B \alpha_1^{A,B} y + \beta_B(\alpha_A \alpha_2^{A,B} x + \beta_A \alpha_3^{A,B} z). \tag{44}$$

By taking $z = \beta_A^{-1} 0$ in (44) (0 is the neutral element of the group $(Q;+)$) we get:

$$\alpha_A(\alpha_B x + \beta_B y) = \alpha_B \alpha_1^{A,B} y + \beta_B(\alpha_A \alpha_2^{A,B} x + \beta_A \alpha_3^{A,B} \beta_A^{-1} 0).$$

The last equality shows that α_A is an antiquasiautomorphism of the group $(Q;+)$ for every $A, B \in \Sigma$. Further, let us take $y = (\alpha_1^{A,B})^{-1} \alpha_B 0$ in (44):

$$\alpha_A(\alpha_B x + \beta_B (\alpha_1^{A,B})^{-1} \alpha_B 0) + \beta_A z = \beta_B(\alpha_A \alpha_2^{A,B} x + \beta_A \alpha_3^{A,B} z).$$

The last equality shows that β_B is a quasiautomorphism of the group $(Q;+)$ for every $A, B \in \Sigma$. Thus,

$$A(x,y) = \overline{\varphi_A} x + c_A + \psi_A y,$$

where $\overline{\varphi_A}$ is an antiautomorphism of $(Q;+)$ and $\psi_A \in Aut(Q;+), c_A \in Q$. Since A is an arbitrary operation from Σ, the algebra $(Q;\Sigma)$ is a mixed type of linearity of the second kind. □

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