Armenian Journal of Mathematics
Volume 12, Number 11, 2020, 1–16

On a family of weighted $\overline{\partial}$-integral representations in the unit disc

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Abstract. For weighted L^p-classess of C^1-functions in the unit disc with weight function of the type $|w|^{2\gamma} \cdot (1 - |w|^{2\rho})^{\alpha}$, we obtain a family of weighted $\overline{\partial}$-integral representations of the type $f = P(f) - T(\overline{\partial} f)$.

Key Words: Smooth Functions in the Unit Disc, Weighted Function Spaces, Weighted $\overline{\partial}$-Integral Representations
Mathematics Subject Classification 2010: 30C40, 30H10, 30H20, 30E20, 32W05

Introduction

Let f be a holomorphic function in the unit disc $\mathbb{D}$ and has (in a certain sense) boundary values on the unit circle $\partial\mathbb{D} = \{w \in C : |w| = 1\}$. Also, denote by σ the Lebesgue measure on $\partial\mathbb{D}$. According to the famous **Cauchy integral formula** written for $\mathbb{D}$,

$$f(z) = \frac{1}{2\pi} \int_{\partial\mathbb{D}} \frac{f(w)}{1 - z\overline{w}} d\sigma(w), \quad z \in \mathbb{D}. \tag{1}$$

A generalization of the Cauchy′s formula for smooth functions (so-called **Cauchy-Green formula**) was established in [1] and for the unit disc can be formulated as follows:

Theorem 1 *If $f \in C^1(\overline{\mathbb{D}})$, then*

$$f(z) = \frac{1}{2\pi} \iint_{\partial\mathbb{D}} \frac{f(w)}{1 - z\overline{w}} d\sigma(w) - \frac{1}{\pi} \iint_{\mathbb{D}} \frac{\partial f(w)/\partial\overline{w}}{w - z} dm(w), \quad z \in \mathbb{D}, \tag{2}$$

where m is two-dimensional Lebesgue measure in the complex plane.

Recall that

$$\frac{\partial f(w)}{\partial \overline{w}} = \frac{1}{2}\left(\frac{\partial f(w)}{\partial x} + i\frac{\partial f(w)}{\partial y}\right) \quad (w = x + iy) \tag{3}$$

is the **Cauchy-Riemann operator**. Apparently, [2] (see also [3]) was the first work, where the values of holomorphic functions inside of a domain were reproduced by integration of functions over the whole domain.

Denote by $H(\mathbb{D})$ the set of all holomorphic functions in the unit disc $\mathbb{D}$.

Theorem 2 *Each function $f \in H(\mathbb{D})$ and satisfying the condition*

$$\iint\limits_{\mathbb{D}} |f(w)|^2 dm(w) < +\infty, \tag{4}$$

has the following integral representation:

$$f(z) = \frac{1}{\pi}\iint\limits_{\mathbb{D}} \frac{f(w)}{(1 - z\cdot\overline{w})^2} dm(w), \quad z \in \mathbb{D}. \tag{5}$$

This result was essentially generalized in [4], [5]: For the spaces $H^p(\alpha) \equiv H(\mathbb{D}) \cap L^p_\alpha(\mathbb{D}), 1 \le p < \infty, \alpha > -1$, of functions f holomorphic in the unit disc $\mathbb{D}$ and satisfying the condition

$$\iint\limits_{\mathbb{D}} |f(w)|^p (1 - |w|^2)^\alpha dm(w) < +\infty, \tag{6}$$

the following assertion is true.

Theorem 3 *Each function $f \in H^p(\alpha)$ has the integral representation*

$$f(z) = \frac{\alpha + 1}{\pi}\iint\limits_{\mathbb{D}} \frac{f(w)(1 - |w|^2)^\alpha}{(1 - z\cdot\overline{w})^{2+\alpha}} dm(w), \quad z \in \mathbb{D}. \tag{7}$$

This result has numerous applications (see, for example, [4], [5]) in the theory of factorization of meromorphic functions in the unit disc as well as to other problems of complex analysis.

A generalization of the formula (7) for smooth functions f (or, equivalently, a weighted version of the formula (2)) has the following form ($Re\beta > -1$):

$$f(z) = \frac{\beta + 1}{\pi}\iint\limits_{\mathbb{D}} \frac{f(w)(1 - |w|^2)^\beta}{(1 - z\overline{w})^{2+\beta}} dm(w) - \frac{1}{\pi}\iint\limits_{\mathbb{D}} \frac{\partial f(w)/\partial \overline{w}}{w - z} \cdot \left(\frac{1 - |w|^2}{1 - z\overline{w}}\right)^{\beta+1} dm(w), \quad z \in \mathbb{D}. \tag{8}$$

This result follows

• from [6] if $f \in C^1(\overline{\mathbb{D}})$;

• from [7] if $f \in C^1(\mathbb{D})$, $grad(f) \in L^1(\mathbb{D})$, and β is real;

• from [8] if $1 \le p < \infty$, $\alpha > -1$, $f \in C^1(\mathbb{D}) \cap L^p_\alpha(\mathbb{D})$, $\partial f(w)/\partial \overline{w} \in L^p_\alpha(\mathbb{D})$, and $\beta = \alpha$;

• from [9] and [10] if $1 \le p < \infty$, $\alpha > -1$, $f \in C^1(\mathbb{D}) \cap L^p_\alpha(\mathbb{D})$, $\partial f(w)/\partial \overline{w} \in L^p_{\alpha+1}(\mathbb{D})$, and $Re\beta \ge \alpha$.

In [11] a further generalization of formula (8) was given by taking a weight function of the type $|w|^{2\gamma} \cdot (1 - |w|^{2\rho})$ instead of $(1-|w|^2)^\alpha$ ($\rho > 0$, $\alpha > -1$ and $\gamma > -1$). The result was a formula of the type

$$f(z) = \iint\limits_{\mathbb{D}} f(w) S_{\alpha,\rho,\gamma}(z;w) \cdot (1-|w|^{2\rho})^\alpha \cdot |w|^{2\gamma} dm(w) - \frac{1}{\pi} \iint\limits_{\mathbb{D}} \frac{\partial f(w)/\partial \overline{w}}{w - z} \cdot Q_{\alpha,\rho,\gamma}(z;w) dm(w), \qquad z \in \mathbb{D}, \quad (9)$$

where $f \in C^1(\overline{\mathbb{D}})$ and $S_{\alpha,\rho,\gamma}(z;w)$ and $Q_{\alpha,\rho,\gamma}(z;w)$ admit integral representations with Mittag-Leffler type kernels.

As it follows from [8], where multidimensional analogue of this result was obtained, the restrictive condition $f \in C^1(\overline{\mathbb{D}})$ in formula (9) can be replaced by

$$f \in C^1(\mathbb{D}) \cap L^p_{\alpha,\rho,\gamma}(\mathbb{D}), \qquad \frac{\partial f(w)}{\partial \overline{w}} \in L^p_{\alpha,\rho,\gamma}(\mathbb{D}), \quad (10)$$

where $\rho > 0$, $\alpha > -1$, $\gamma > -1$, and the spaces $L^p_{\alpha,\rho,\gamma}(\mathbb{D})$ are naturally generated by the "norm"

$$M^p_{\alpha,\rho,\gamma}(f) = \iint\limits_{\mathbb{D}} |f(w)|^p \cdot (1-|w|^{2\rho})^\alpha \cdot |w|^{2\gamma} dm(w). \quad (11)$$

Our goal is to obtain (for fixed $\alpha > -1$, $\gamma > -1$) a family of integral representations of type (9) with kernels $S_{\beta,\rho,\varphi}(z;w)$ and $Q_{\beta,\rho,\varphi}(z;w)$ depending on complex parameters β and φ with $Re\beta \ge \alpha$, $Re\varphi \ge \gamma$ (for holomorphic f that was done in [12]). In comparison with [11] and [8], we write out the kernels $Q_{\beta,\rho,\varphi}(z;w)$ in a series form which admits to specify their certain properties. Moreover, we weaken the second growth condition in (10) by assuming that

$$\frac{\partial f(w)}{\partial \overline{w}} \in L^p_{\alpha+1,\rho,\gamma}(\mathbb{D}). \quad (12)$$

1 The kernel $Q_{\beta,\rho,\varphi}(z;w)$ and its properties

In what follows we assume that $Re\beta > -1$, $Re\varphi > -1$, $\rho > 0$, and $\mu = (\varphi+1)/\rho$. For arbitrary $z \in \mathbb{D}$, $w \in \overline{\mathbb{D}}$, put

$$S_{\beta,\rho,\varphi}(z;w) = \frac{\rho}{\pi \cdot \Gamma(\beta+1)} \cdot \sum_{k=0}^{\infty} \frac{\Gamma(\mu+\beta+1+\frac{k}{\rho})}{\Gamma(\mu+\frac{k}{\rho})} \cdot z^k \cdot \overline{w}^k. \tag{13}$$

This kernel was written out in [12] to generalize (9) for holomorphic functions. To obtain the corresponding kernel $Q_{\beta,\rho,\varphi}$ (i.e., to generalize (9) for C^1–functions) we use the relation between Q and S obtained in [11] (formula (2.11)):

$$Q_{\beta,\rho,\varphi}(z;w) = \iint\limits_{\mathbb{D}} S_{\beta,\rho,\varphi}(z;\zeta) \cdot \frac{\zeta - z}{\zeta - w} \cdot \left(1-|\zeta|^{2\rho}\right)^{\beta} \cdot |\zeta|^{2\varphi} dm(\zeta). \tag{14}$$

Using the expansion (13), as well as the residue theorem, we have arrived at the following formula for arbitrary $z \in D$ (we omit some technical details):

$$Q_{\beta,\rho,\varphi}(z;w) = 1 + \frac{(z-w)\rho}{w\Gamma(\beta+1)} \sum_{k=0}^{\infty} \frac{\Gamma\left(\mu+\beta+1+\frac{k}{\rho}\right)}{\Gamma\left(\mu+\frac{k}{\rho}\right)} \frac{z^k}{w^k} \int\limits_0^{|w|^2} \left(1-t^{\rho}\right)^{\beta} t^{\varphi+k} dt$$

$$\equiv 1 + \frac{z-w}{w \cdot \Gamma(\beta+1)} \sum_{k=0}^{\infty} \frac{\Gamma\left(\mu+\beta+1+\frac{k}{\rho}\right)}{\Gamma\left(\mu+\frac{k}{\rho}\right)} \cdot \frac{z^k}{w^k} \cdot \int\limits_0^{|w|^{2\rho}} (1-x)^{\beta} x^{\mu+\frac{k}{\rho}-1} dx, \tag{15}$$

where $w \in \overline{\mathbb{D}} \backslash \{0\}$, and

$$Q_{\beta,\rho,\varphi}(z;0) \equiv 1. \tag{16}$$

To simplify notation, in what follows, we will use the notations Q and S instead of $Q_{\beta,\rho,\varphi}$ and $S_{\beta,\rho,\varphi}$, respectively.

To check whether the kernel Q is well-defined, we estimate the corresponding series in (15) by absolutely convergent numerical series (the so-called majorated convergence). We have

$$\sum_{k=0}^{\infty} \left| \frac{\Gamma\left(\mu+\beta+1+\frac{k}{\rho}\right)}{\Gamma\left(\mu+\frac{k}{\rho}\right)} \cdot \frac{z^k}{w^k} \cdot \int\limits_0^{|w|^2} \left(1-t^{\rho}\right)^{\beta} t^{\varphi+k} dt \right|$$

$$\leq \sum_{k=0}^{\infty} \frac{\Gamma\left(Re\mu+Re\beta+1+\frac{k}{\rho}\right)}{\Gamma\left(Re\mu+\frac{k}{\rho}\right)} \cdot \frac{|z|^k}{|w|^k} \cdot \left| \int\limits_0^{|w|^2} \left(1-t^{\rho}\right)^{\beta} t^{\varphi+k} dt \right|$$

$$\leq const(\beta,\rho,\varphi)\sum_{k=0}^{\infty}\frac{\Gamma\left(k+Re\beta+2\right)}{\Gamma(k+1)}\cdot\frac{|z|^k}{|w|^k}\cdot|w|^{2k}\cdot\int\limits_0^{|w|^2}\left(1-t^{\rho}\right)^{Re\beta}t^{Re\varphi}dt$$

$$\leq const(\beta,\rho,\varphi)\sum_{k=0}^{\infty}\frac{\Gamma\left(k+Re\beta+2\right)}{\Gamma(k+1)}\cdot\frac{|z|^k}{|w|^k}\cdot|w|^{2k}\cdot\int\limits_0^{1}\left(1-t^{\rho}\right)^{Re\beta}t^{Re\varphi}dt$$

$$\leq const(\beta,\rho,\varphi)\sum_{k=0}^{\infty}\frac{\Gamma\left(k+Re\beta+2\right)}{\Gamma(k+1)}\cdot|z|^k\cdot|w|^k$$

$$\leq const(\beta,\rho,\varphi)\sum_{k=0}^{\infty}\frac{\Gamma\left(k+Re\beta+2\right)}{\Gamma(k+1)}\cdot|z|^k=const(\beta,\rho,\varphi)\frac{1}{(1-|z|)^{Re\beta+2}}.$$

Hence, the following assertion is true.

Proposition 1 *The kernel $Q(z;w)$ is well-defined for $z\in\mathbb{D}$ and $w\in\overline{\mathbb{D}}$. Moreover, $Q(z;w)$ is continuous in $\overline{\mathbb{D}}\backslash\{0\}$ for fixed z and holomorphic in $\mathbb{D}$ for fixed w.*

Remark 1 *In the estimation above we used the following consequence of the Stirling's formula:*

$$\frac{|\Gamma(\mu+R)|}{|\Gamma(\nu+R)|}\asymp R^{Re\mu-Re\nu},\quad R\to+\infty. \tag{17}$$

Proposition 2 *Suppose* $0<|w|\leq\dfrac{1}{2}$. *Then*

$$|Q(z;w)-Q(z;0)|\equiv|Q(z;w)-1|=const(\beta,\rho,\varphi,z)\cdot\begin{cases}|w|^{2Re\varphi+1}, & z\neq0,\\ |w|^{2Re\varphi+2}, & z=0.\end{cases}$$

Proof. **Case 1:** Let $z\neq0$. Then

$$|Q(z;w)-1|$$
$$\leq const(\beta,\rho,\varphi)\sum_{k=0}^{\infty}\frac{\Gamma\left(k+Re\beta+2\right)}{\Gamma(k+1)}\cdot\frac{|z|^k}{|w|^{k+1}}\cdot\int\limits_0^{|w|^2}\left(1-t^{\rho}\right)^{Re\beta}t^{Re\varphi+k}dt.$$

Since $0\leq t\leq\dfrac{1}{4}$, we have

$$0\leq t^{\rho}\leq\frac{1}{4^{\rho}}\qquad\text{and}\qquad 1-\frac{1}{4^{\rho}}\leq1-t^{\rho}\leq1.$$

Hence,

$$\left(1-t^{\rho}\right)^{Re\beta}\leq\begin{cases}1, & \text{if } Re\beta>0,\\ \left(1-\frac{1}{4^{\rho}}\right)^{Re\beta}, & \text{if } -1<Re\beta\leq0.\end{cases}$$

Therefore,

$$
\begin{aligned}
|Q(z;w)-1| &\le const(\beta,\rho,\varphi)\sum_{k=0}^{\infty}\frac{\Gamma\left(k+Re\beta+2\right)}{\Gamma(k+1)}\cdot\frac{|z|^k}{|w|^{k+1}}\cdot\int\limits_0^{|w|^2} t^{Re\varphi+k}dt \\
&\le const(\beta,\rho,\varphi)\sum_{k=0}^{\infty}\frac{\Gamma\left(k+Re\beta+2\right)}{\Gamma(k+1)}\cdot\frac{|z|^k}{|w|^{k+1}}\cdot\frac{|w|^{2(Re\varphi+k+1)}}{Re\varphi+k+1} \\
&\le const(\beta,\rho,\varphi)\sum_{k=0}^{\infty}\frac{\Gamma\left(k+Re\beta+2\right)}{\Gamma(k+1)(k+1)}\cdot|z|^k\cdot|w|^k\cdot|w|^{2Re\varphi+1} \\
&= const(\beta,\rho,\varphi)\sum_{k=0}^{\infty}\frac{\Gamma\left(k+Re\beta+2\right)}{\Gamma(k+2)}\cdot|z|^k\cdot|w|^k\cdot|w|^{2Re\varphi+1} \\
&\le const(\beta,\rho,\varphi)\sum_{k=0}^{\infty}k^{Re\beta}\cdot|z|^k\cdot|w|^k\cdot|w|^{2Re\varphi+1} \\
&\le const(\beta,\rho,\varphi)\sum_{k=0}^{\infty}\frac{\Gamma\left(k+Re\beta+1\right)}{\Gamma(k+1)}\cdot|z|^k\cdot|w|^k\cdot|w|^{2Re\varphi+1} \\
&\le const(\beta,\rho,\varphi)|w|^{2Re\varphi+1}\cdot\frac{1}{(1-|z|)^{Re\beta+1}}.
\end{aligned}
$$

Case 2: Let $z=0$. It is easy to see that in this case $(z-w)/w=-1$, that is why the power of $|w|$ in our estimate increases by 1. □

Corollary 1 *If $z=0$, then $Q(z;w)$ is continuous at the point $w=0$ (therefore, in $\overline{\mathbb{D}}$); and if $z\neq 0$, then $Q(z;w)$ is continuous at $w=0$ if and only if $Re\varphi>-1/2$.*

Lemma 1 *If $|z|<|w|\le 1$, then*

$$
\frac{z-w}{w\Gamma(\beta+1)}\sum_{k=0}^{\infty}\frac{\Gamma\left(\mu+\beta+1+\frac{k}{\rho}\right)}{\Gamma\left(\mu+\frac{k}{\rho}\right)}\cdot\frac{z^k}{w^k}\cdot\int\limits_0^1(1-x)^{\beta}x^{\mu+\frac{k}{\rho}-1}dx=-1.
$$

Proof. Note that

$$
\int\limits_0^1(1-x)^{\beta}x^{\mu+\frac{k}{\rho}-1}dx=B\left(\beta+1,\mu+\frac{k}{\rho}\right)=\frac{\Gamma(\beta+1)\Gamma\left(\mu+\frac{k}{\rho}\right)}{\Gamma\left(\mu+\beta+1+\frac{k}{\rho}\right)}.
$$

Hence

$$\frac{z-w}{w\Gamma(\beta+1)}\sum_{k=0}^{\infty}\frac{\Gamma\left(\mu+\beta+1+\frac{k}{\rho}\right)}{\Gamma\left(\mu+\frac{k}{\rho}\right)}\cdot\frac{z^k}{w^k}\cdot\int\limits_0^1(1-x)^{\beta}x^{\mu+\frac{k}{\rho}-1}dx$$

$$=\frac{z-w}{w\Gamma(\beta+1)}\sum_{k=0}^{\infty}\frac{\Gamma\left(\mu+\beta+1+\frac{k}{\rho}\right)}{\Gamma\left(\mu+\frac{k}{\rho}\right)}\cdot\frac{z^k}{w^k}\cdot\frac{\Gamma(\beta+1)\Gamma\left(\mu+\frac{k}{\rho}\right)}{\Gamma\left(\mu+\beta+1+\frac{k}{\rho}\right)}$$

$$=\frac{z-w}{w}\sum_{k=0}^{\infty}\frac{z^k}{w^k}=\frac{z-w}{w}\cdot\frac{1}{1-\frac{z}{w}}=-1.$$

□

Corollary 2 *If $z\in\mathbb{D}$ and $w\in\partial\mathbb{D}$, then $Q(z;w)=0$.*

The following assertion is evident.

Proposition 3 *If $w=z\in\mathbb{D}$, then $Q(z;w)\equiv Q(z;z)\equiv 1$.*

Proposition 4 *Let $(1+|z|)/2\le|w|\le 1$, $K\subset\mathbb{D}$ be a compact and let $z\in K$. Then*

$$|Q(z;w)|\le const(\beta,\rho,\varphi)\frac{\left(1-|w|^{2\rho}\right)^{Re\beta+1}}{\left(1-|z|\right)^{Re\beta+2}}$$
$$\le const(\beta,\rho,\varphi,K)\cdot\left(1-|w|^{2\rho}\right)^{Re\beta+1}. \quad (18)$$

Proof. From $(1+|z|)/2\le|w|\le 1$ it follows that

$$|z|<|w|,\qquad |w|\ge\frac{1}{2},\qquad \text{and}\qquad |w|-|z|\ge\frac{1-|z|}{2}.$$

According to Lemma 1 we have:

$$Q(z;w)=1+\frac{z-w}{w\Gamma(\beta+1)}\sum_{k=0}^{\infty}\frac{\Gamma\left(\mu+\beta+1+\frac{k}{\rho}\right)}{\Gamma\left(\mu+\frac{k}{\rho}\right)}\cdot\frac{z^k}{w^k}$$
$$\cdot\left(\int\limits_0^1(1-x)^{\beta}x^{\mu+\frac{k}{\rho}-1}dx-\int\limits_{|w|^{2\rho}}^1(1-x)^{\beta}x^{\mu+\frac{k}{\rho}-1}dx\right)$$
$$=\frac{w-z}{w\Gamma(\beta+1)}\sum_{k=0}^{\infty}\frac{\Gamma\left(\mu+\beta+1+\frac{k}{\rho}\right)}{\Gamma\left(\mu+\frac{k}{\rho}\right)}\cdot\frac{z^k}{w^k}\cdot\int\limits_{|w|^{2\rho}}^1(1-x)^{\beta}x^{\mu+\frac{k}{\rho}-1}dx.$$

Hence,

$$|Q(z;w)| \leq \frac{const(\beta,\rho,\varphi)}{|w|}$$
$$\cdot \sum_{k=0}^{\infty} \frac{\Gamma\left(k+Re\beta+2\right)}{\Gamma\left(k+1\right)} \cdot \frac{|z|^k}{|w|^k} \cdot \int\limits_{|w|^{2\rho}}^{1} (1-x)^{Re\beta} x^{Re\mu+\frac{k}{\rho}-1} dx.$$

If $Re\mu + \dfrac{k}{\rho} - 1 \geq 0$, then $x^{Re\mu+\frac{k}{\rho}-1} \leq 1$, otherwise,

$$x^{Re\mu+\frac{k}{\rho}-1} \leq |w|^{2\rho\left(Re\mu+\frac{k}{\rho}-1\right)} \leq |w|^{2\rho(Re\mu-1)} \leq \left(\frac{1}{2}\right)^{2\rho(Re\mu-1)} = const(\rho,\varphi).$$

Thus, we can write

$$|Q(z;w)| \leq \frac{const(\beta,\rho,\varphi)}{|w|} \sum_{k=0}^{\infty} \frac{\Gamma\left(k+Re\beta+2\right)}{\Gamma\left(k+1\right)} \cdot \frac{|z|^k}{|w|^k} \cdot \int\limits_{|w|^{2\rho}}^{1} (1-x)^{Re\beta} dx$$
$$= const(\beta,\rho,\varphi) \sum_{k=0}^{\infty} \frac{\Gamma\left(k+Re\beta+2\right)}{\Gamma\left(k+1\right)} \cdot \frac{|z|^k}{|w|^k} \cdot \left(1-|w|^{2\rho}\right)^{Re\beta+1}$$
$$= const(\beta,\rho,\varphi) \frac{\left(1-|w|^{2\rho}\right)^{Re\beta+1}}{\left(1-\frac{|z|}{|w|}\right)^{Re\beta+2}} \leq const(\beta,\rho,\varphi) \frac{\left(1-|w|^{2\rho}\right)^{Re\beta+1}}{\left(|w|-|z|\right)^{Re\beta+2}}$$
$$\leq const(\beta,\rho,\varphi) \frac{\left(1-|w|^{2\rho}\right)^{Re\beta+1}}{\left(1-|z|\right)^{Re\beta+2}} \leq const(\beta,\rho,\varphi,K) \left(1-|w|^{2\rho}\right)^{Re\beta+1}.$$

□

Remark 2 *For real β and φ, Propositions 2 and 4 were formulated in [11], where schemes of proofs were also given.*

Proposition 5 *For a fixed $z \in \mathbb{D}$, $Q(z;w) \in C^1\left(\mathbb{D}\backslash\{0\}\right)$.*

Proof. For $w \in \mathbb{D}\backslash\{0\}$, we formally have:

$$\frac{\partial Q(z;w)}{\partial \overline{w}} = \frac{(z-w)\rho}{\Gamma(\beta+1)}$$
$$\cdot \sum_{k=0}^{\infty} \frac{\Gamma\left(\mu+\beta+1+\frac{k}{\rho}\right)}{\Gamma\left(\mu+\frac{k}{\rho}\right)} \cdot \frac{z^k}{w^{k+1}} \cdot \left(1-|w|^{2\rho}\right)^{\beta} \cdot |w|^{2(\varphi+k)} \cdot w$$
$$= \frac{(z-w)\rho}{\Gamma(\beta+1)} \cdot \left(1-|w|^{2\rho}\right)^{\beta} \cdot |w|^{2\varphi} \sum_{k=0}^{\infty} \frac{\Gamma\left(\mu+\beta+1+\frac{k}{\rho}\right)}{\Gamma\left(\mu+\frac{k}{\rho}\right)} \cdot z^k \cdot \overline{w}^k.$$

Making a majorating estimation for the expression above when $0 < \varepsilon \leq |w| \leq 1 - \varepsilon < 1$ we obtain

$$\left|\frac{\partial Q(z;w)}{\partial \overline{w}}\right| \leq const(\beta,\rho,\varphi,\varepsilon) \cdot \sum_{k=0}^{\infty} \frac{\Gamma\left(k + Re\beta + 2\right)}{\Gamma\left(k+1\right)} \cdot |z|^k \cdot |\overline{w}|^k$$

$$\leq const(\beta,\rho,\varphi,\varepsilon) \cdot \sum_{k=0}^{\infty} \frac{\Gamma\left(k + Re\beta + 2\right)}{\Gamma\left(k+1\right)} \cdot |z|^k$$

$$= const(\beta,\rho,\varphi,\varepsilon) \cdot \frac{1}{\left(1-|z|\right)^{Re\beta+2}}.$$

Now consider the formal expression

$$\frac{\partial Q(z;w)}{\partial w} = A_1 + A_2 + A_3,$$

where

$$A_1 = -\frac{\rho}{\Gamma(\beta+1)} \sum_{k=0}^{\infty} \frac{\Gamma\left(\mu+\beta+1+\frac{k}{\rho}\right)}{\Gamma\left(\mu+\frac{k}{\rho}\right)} \cdot \frac{z^k}{w^{k+1}} \cdot \int_0^{|w|^2} \left(1-t^{\rho}\right)^{\beta} t^{\varphi+k} dt,$$

$$A_2 = \frac{(z-w)\rho}{\Gamma(\beta+1)} \sum_{k=0}^{\infty} \frac{\Gamma\left(\mu+\beta+1+\frac{k}{\rho}\right)}{\Gamma\left(\mu+\frac{k}{\rho}\right)} \cdot \frac{z^k}{w^{k+2}}(-k-1) \int_0^{|w|^2} \left(1-t^{\rho}\right)^{\beta} t^{\varphi+k} dt,$$

$$A_3 = \frac{(z-w)\rho}{\Gamma(\beta+1)} \sum_{k=0}^{\infty} \frac{\Gamma\left(\mu+\beta+1+\frac{k}{\rho}\right)}{\Gamma\left(\mu+\frac{k}{\rho}\right)} \cdot \frac{z^k}{w^{k+1}} \cdot \left(1-|w|^{2\rho}\right)^{\beta} \cdot |w|^{2(\varphi+k)} \cdot \overline{w}.$$

Let us majorate the series above under the same condition $0 < \varepsilon \leq |w| \leq 1 - \varepsilon < 1$. We can write

$$|A_1| \leq \frac{const(\beta,\rho,\varphi)}{|w|} \sum_{k=0}^{\infty} \frac{\Gamma\left(k + Re\beta + 2\right)}{\Gamma\left(k+1\right)} \cdot |z|^k \cdot |\overline{w}|^k$$

$$\leq const(\beta,\rho,\varphi,\varepsilon) \sum_{k=0}^{\infty} \frac{\Gamma\left(k + Re\beta + 2\right)}{\Gamma\left(k+1\right)} \cdot |z|^k < +\infty,$$

$$|A_2| \leq \frac{const(\beta,\rho,\varphi)}{|w|^2} \sum_{k=0}^{\infty} (k+1)^{Re\beta+2} \cdot |z|^k \cdot |w|^k$$

$$\leq const(\beta,\rho,\varphi,\varepsilon) \sum_{k=0}^{\infty} \frac{\Gamma\left(k + Re\beta + 3\right)}{\Gamma\left(k+1\right)} \cdot |z|^k < +\infty,$$

$$|A_3| \le const(\beta,\rho,\varphi)\cdot\left(1-|w|^{2\rho}\right)^{\beta}\cdot|w|^{2Re\varphi}\sum_{k=0}^{\infty}\frac{\Gamma\left(k+Re\beta+2\right)}{\Gamma\left(k+1\right)}\cdot|z|^{k}\cdot|\overline{w}|^{k}$$
$$\le const(\beta,\rho,\varphi,\varepsilon)\sum_{k=0}^{\infty}\frac{\Gamma\left(k+Re\beta+2\right)}{\Gamma\left(k+1\right)}\cdot|z|^{k} < +\infty.$$

Thus, we have shown that for a fixed $z \in \mathbb{D}$ the derivatives $\partial Q(z;w)/\partial\overline{w}$ and $\partial Q(z;w)/\partial w$ exist and are continuous in $\mathbb{D}\backslash\{0\}$. Hence, the proposition is proved. □

Proposition 6 *For $z \in \mathbb{D}$ and $w \in \mathbb{D}\backslash\{0\}$, the following relation holds:*

$$\frac{\partial Q(z;w)}{\partial\overline{w}} = S(z;w)\cdot\pi\cdot(z-w)\cdot|w|^{2\varphi}\cdot(1-|w|^{2\varphi})^{\beta}. \tag{19}$$

Proof. Indeed,

$$\frac{\partial Q(z;w)}{\partial\overline{w}} = \frac{(z-w)\rho}{w\Gamma(\beta+1)}\sum_{k=0}^{\infty}\frac{\Gamma\left(\mu+\beta+1+\frac{k}{\rho}\right)}{\Gamma\left(\mu+\frac{k}{\rho}\right)}\frac{z^k}{w^k}\left(1-(w\overline{w})^{\rho}\right)^{\beta}(w\overline{w})^{\varphi+k}w$$
$$= \pi(z-w)|w|^{2\varphi}\left(1-(w\cdot\overline{w})^{\rho}\right)^{\beta}\frac{\rho}{\pi\Gamma(\beta+1)}\sum_{k=0}^{\infty}\frac{\Gamma\left(\mu+\beta+1+\frac{k}{\rho}\right)}{\Gamma\left(\mu+\frac{k}{\rho}\right)}z^k\overline{w}^k$$
$$= S(z;w)\cdot\pi\cdot(z-w)\cdot|w|^{2\varphi}\cdot(1-|w|^{2\rho})^{\beta}.$$

□

2 The main integral representation

In what follows, we assume that $\alpha > -1$, $\gamma > -1$, $\rho > 0$, $Re\beta \ge \alpha$, $Re\varphi \ge \gamma$, and $\mu = (\varphi+1)/\rho$.

Recall that the "norms" (11) generate the corresponding spaces $L^p_{\alpha,\rho,\gamma}(\mathbb{D})$, where $1 \le p < +\infty$. It is easy to check that $L^p_{\alpha,\rho,\gamma}(\mathbb{D}) \subset L^1_{\alpha,\rho,\gamma}(\mathbb{D})$.

Assume that $f \in L^1_{\alpha,\rho,\gamma}(\mathbb{D}) \cap C(\mathbb{D})$ and put

$$M(t) \equiv \int_0^{2\pi}\left|f\left(te^{i\theta}\right)\right|d\theta, \quad t \in [0,1). \tag{20}$$

Evidently, $M(t)$ is continuous with respect to t.

In view of (20), the condition $f \in L^1_{\alpha,\rho,\gamma}(\mathbb{D})$ can be written as

$$\int_0^1 M(t)\left(1-t^{2\rho}\right)^{\alpha}t^{2\gamma+1}dt < +\infty, \tag{21}$$

and since $f \in C(\mathbb{D})$, (21) is equivalent to the following condition

$$\int_0^1 M(t)\left(1-t^{2\rho}\right)^{\alpha} dt < +\infty. \tag{22}$$

Lemma 2 *If $f \in L^1_{\alpha,\rho,\gamma}(\mathbb{D}) \cap C(\mathbb{D})$, then there exists a sequence $\{t_k\}_{k=1}^{\infty} \subset (0;1)$ such that $t_k \to 1$ as $k \to \infty$ and*

$$\lim_{k\to\infty} M(t_k)\left(1-t_k^{2\rho}\right)^{\alpha+1} = 0.$$

Proof. Assume the opposite. Suppose there exists $\varepsilon > 0$ such that $M(t)\,(1-t^{2\rho})^{\alpha+1} \geq \varepsilon$ for every $t \in (1-\delta, 1)$. From here it follows that

$$M(t)\left(1-t^{2\rho}\right)^{\alpha} \geq \frac{\varepsilon}{1-t^{2\rho}}, \qquad t \in (1-\delta, 1).$$

But the inequality above contradicts to (22). □

Theorem 4 *Let $1 \leq p < +\infty$, $\alpha > -1, \gamma > -1, \rho > 0, Re\beta \geq \alpha$, and $Re\varphi \geq \gamma$. Also, let*

$$f \in L^p_{\alpha,\rho,\gamma}(\mathbb{D}) \cap C^1(\mathbb{D}) \tag{23}$$

and

$$\frac{\partial f(w)}{\partial \overline{w}} \in L^p_{\alpha+1,\rho,\gamma}(\mathbb{D}). \tag{24}$$

Then for every $z \in \mathbb{D}$

$$f(z) = \iint_{\mathbb{D}} f(w) S(z;w)\left(1-|w|^{2\rho}\right)^{\beta} |w|^{2\varphi} dm(w) - \frac{1}{\pi}\iint_{\mathbb{D}} \frac{\partial f(w)/\partial\overline{w}}{w-z} Q(z;w) dm(w). \tag{25}$$

Proof. It is enough to prove the assertion only in the case $p = 1$. Let us consider the differential form

$$\tau_z(w) = \frac{1}{2\pi i} \cdot \frac{f(w)Q(z;w)}{w-z} dw, \quad w \in \mathbb{D}\backslash\{0\}. \tag{26}$$

Case 1: $z \neq 0$. Denote (see Figure 1)

$$D_1 = \{|w| \leq t_k\}, \quad k = 1, 2, 3, \ldots,$$

where the sequence $\{t_k\}_1^{\infty}$ is chosen as in Lemma 2,

$$D_2 = \{|w| \leq \varepsilon_1\}, \qquad D_3 = \{|w-z| \leq \varepsilon_2\}, \qquad G = D_1 \backslash \{D_2 \cup D_3\}.$$

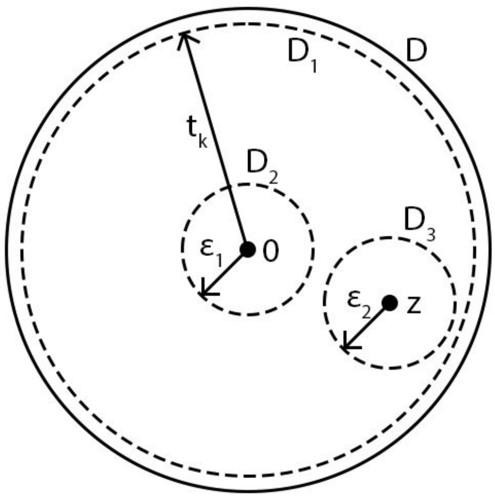

Figure 1: $z \neq 0$ case

Using Stokes formula we get

$$\frac{1}{2\pi i} \cdot \int\limits_{|w|=t_k} \frac{f(w)Q(z;w)}{w-z} dw$$

$$- \frac{1}{2\pi i} \cdot \int\limits_{|w-z|=\varepsilon_2} \frac{f(w)Q(z;w)}{w-z} dw - \frac{1}{2\pi i} \cdot \int\limits_{|w|=\varepsilon_1} \frac{f(w)Q(z;w)}{w-z} dw$$

$$= \frac{1}{2\pi i} \iint\limits_G \frac{\partial f(w)/\partial \overline{w}}{w-z} Q(z;w) d\overline{w} \wedge dw + \frac{1}{2\pi i} \iint\limits_G \frac{f(w)}{w-z} \frac{\partial Q(z;w)}{\partial \overline{w}} d\overline{w} \wedge dw,$$

which can be symbolically written as follows

$$I_1 - I_2 - I_3 = I_4 + I_5.$$

Now let us estimate I_i, $i = 1, ..., 5$, as $t_k \to 1, \varepsilon_1 \to 0, \varepsilon_2 \to 0$.

We can suppose that $(1 + |z|)/2 < t_k < 1$. Then

$$|I_1| \le \frac{1}{2\pi} \int\limits_0^{2\pi} \frac{\left|f\left(t_k e^{i\theta}\right)\right| \cdot \left|Q\left(z; t_k e^{i\theta}\right)\right|}{|t_k e^{i\theta} - z|} \cdot t_k d\theta$$

$$\le \frac{const(\beta, \rho, \varphi, z)}{\pi} \int\limits_0^{2\pi} \frac{\left|f\left(t_k e^{i\theta}\right)\right| \cdot \left(1 - t_k^{2\rho}\right)^{\alpha+1}}{1 - |z|} d\theta$$

$$= const(\beta, \rho, \varphi, z) \left(1 - t_k^{2\rho}\right)^{\alpha+1} \int\limits_0^{2\pi} \left|f\left(t_k e^{i\theta}\right)\right| d\theta$$

$$\equiv const(\beta, \rho, \varphi, z) \left(1 - t_k^{2\rho}\right)^{\alpha+1} \cdot M(t_k).$$

But from Lemma 2 it follows that $\left(1-t_k^{2\rho}\right)^{\alpha+1}\cdot M(t_k)\to 0$ as $t_k\to 1$. Hence, $I_1\to 0$ as $t_k\to 1$.

Since $f(w)\cdot Q(z;w)$ is continuous in the neighborhood of z,

$$I_2\to f(z)\cdot Q(z;z)\equiv f(z)\quad \text{as } \varepsilon_2\to 0.$$

Further,

$$|I_3|\le \frac{1}{2\pi}\cdot\frac{2}{|z|}\cdot\max_{|w|\le\frac{1}{2}}|f(w)|\cdot\left(1+const(\beta,\rho,\varphi,z)\cdot\varepsilon_1^{2Re\varphi+1}\right)\cdot 2\pi\varepsilon_1\to 0$$

as $\varepsilon_1\to 0$.

Thus, in view of (19),

$$\begin{aligned}-f(z)&=\frac{1}{\pi}\cdot\iint\limits_{\mathbb{D}}\frac{\partial f(w)/\partial\overline{w}}{w-z}Q(z;w)dm(w)\\&+\frac{1}{\pi}\cdot\iint\limits_{\mathbb{D}}\frac{f(w)}{w-z}\cdot\pi\cdot(z-w)\cdot\left(1-|w|^{2\rho}\right)^{\beta}\cdot|w|^{2\varphi}\cdot S(z;w)dm(w),\end{aligned}$$

or, equivalently,

$$\begin{aligned}f(z)=\iint\limits_{\mathbb{D}}f(w)\cdot\left(1-|w|^{2\rho}\right)^{\beta}\cdot|w|^{2\varphi}\cdot S(z;w)dm(w)\\-\frac{1}{\pi}\cdot\iint\limits_{\mathbb{D}}\frac{\partial f(w)/\partial\overline{w}}{w-z}Q(z;w)dm(w).\end{aligned}$$

To make the above passage to the limit correct, we need to make sure that both plane integrals converge. For I_5 the correctness follows from [12]. Let us consider I_4. It is easy to see that

$$\frac{\partial f(w)/\partial\overline{w}}{w-z}\cdot Q(z;w)$$

is integrable in the neighborhood of z (since $1/(w-z)$ has integrable singularity) and in the neighborhood of 0 (in view of Proposition 2). To check the integrability near the boundary of the unit disc, put $D^\star=\left\{\frac{1+|z|}{2}<|w|<1\right\}$. Then, due to the estimate (18), we have

$$\begin{aligned}&\iint\limits_{D^\star}\frac{|\partial f(w)/\partial\overline{w}|}{|w-z|}\cdot|Q(z;w)|dm(w)\\&\le\frac{2const(\beta,\rho,\varphi)}{(1-|z|)^{Re\beta+3}}\iint\limits_{D^\star}\left|\frac{\partial f(w)}{\partial\overline{w}}\right|\left(1-|w|^{2\rho}\right)^{Re\beta+1}dm(w)\\&\le\frac{2const(\beta,\rho,\varphi)}{(1-|z|)^{Re\beta+3}}\iint\limits_{D^\star}\left|\frac{\partial f(w)}{\partial\overline{w}}\right|\left(1-|w|^{2\rho}\right)^{\alpha+1}dm(w)<+\infty.\end{aligned}$$

Case 2: $z = 0$. Denote (see Figure 2)

$$D_1 = \{|w| \le t_k\}, \quad k = 1, 2, 3, \dots,$$

where the sequence $\{t_k\}_1^\infty$ is chosen as in Lemma 2,

$$D_2 = \{|w| \le \varepsilon\}, \qquad G = D_1 \backslash D_2.$$

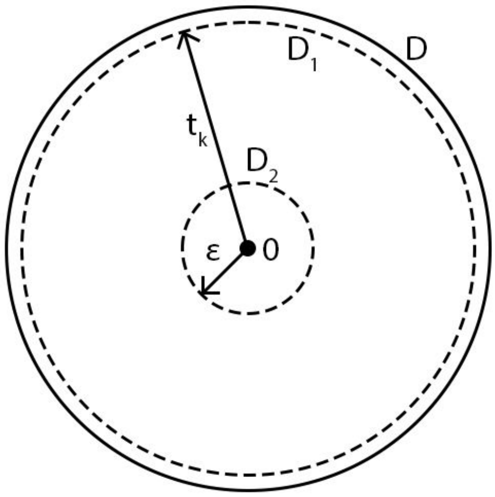

Figure 2: $z = 0$ case

Again, using Stokes formula we get

$$\frac{1}{2\pi i} \cdot \int\limits_{|w|=t_k} \frac{f(w)Q(0;w)}{w} dw - \frac{1}{2\pi i} \cdot \int\limits_{|w|=\varepsilon} \frac{f(w)Q(0;w)}{w} dw$$
$$= \frac{1}{\pi} \cdot \iint\limits_G \frac{\partial f(w)/\partial \overline{w}}{w} \cdot Q(0;w) dm(w) + \frac{1}{\pi} \cdot \iint\limits_G \frac{f(w)}{w} \cdot \frac{\partial Q(0;w)}{\partial \overline{w}} dm(w),$$

or, symbolically

$$J_1 - J_2 = J_3 + J_4.$$

In the same way as in Case 1, $J_1 \to 0$ as $t_k \to 1$. Further,

$$J_2 = \frac{1}{2\pi i} \cdot \int\limits_{|w|=\varepsilon} \frac{f(w)}{w} (Q(0;w) - 1) dw + \frac{1}{2\pi i} \cdot \int\limits_{|w|=\varepsilon} \frac{f(w)}{w} dw \equiv B_1 + B_2.$$

Here

$$|B_1| \le \frac{1}{2\pi} \cdot \frac{\max\limits_{|w| \le \frac{1}{2}} |f(w)| \cdot \varepsilon^{2Re\varphi + 2}}{\varepsilon} \cdot 2\pi\varepsilon \to 0$$

and

$$B_2 \to f(0)$$

as $\varepsilon \to 0$.

Finally, we need to make sure that integrals J_3 and J_4 are convergent. For the convergence of J_4 see [12]. The estimation of integral J_3 near the boundary of $\mathbb{D}$ can be done in the same way we did it in Case 1. In the neighborhood of 0, in view of Proposition 2, we have

$$|J_3| \leq \iint\limits_{|w|\leq\frac{1}{2}} \max_{|w|\leq\frac{1}{2}} \left|\frac{\partial f(w)}{\partial \overline{w}}\right| \cdot \frac{1 + const(\beta,\rho,\varphi)\cdot|w|^{2Re\varphi+2}}{|w|} dm(w)$$

$$\leq \max_{|w|\leq\frac{1}{2}} \left|\frac{\partial f(w)}{\partial \overline{w}}\right| \cdot 2\pi \cdot \int\limits_0^{\frac{1}{2}} \frac{1 + const(\beta,\rho,\varphi)\cdot r^{2Re\varphi+2}}{r} \cdot rdr < +\infty.$$

This concludes the proof of the theorem. □

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Please, cite to this paper as published in
Armen. J. Math., V. **12**, N. 11(2020), pp. 1–16