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On the regularity of weak solutions to refractor problem

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Abstract

In this note we derive the Monge-Ampère type equation in Euclidian coordinates describing the refraction phenomena of perfect lens. This simplifies the regularity issues of the weak solutions on the problem.

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1 Introduction and main result

It is well-known that ellipse and hyperbola have simple refraction properties, namely if rays of light diverge from one focus, then after refraction they pass parallel to the major axis [4]. If the ellipse (resp. hyperbola) represents the boundary separating two medias, with refractive indices n_1, n_2 then according to refraction law

$$n_1 \sin\alpha = n_2 \sin\beta,$$

where α and β are the angles between normal and respectively the ray before and after refraction. Let $k = n_1/n_2$, then one can verify that $k = 1/\varepsilon$, where ε is the eccentricity of ellipse (resp. hyperbola) [4]. These properties are limiting cases of solutions to more general problems of determining the surface required to refract rays of light diverging from one point and after refraction covering a given set of directions on the unit sphere. More precisely let us assume we are given two sets Ω, Ω^* on unit sphere centered at origin, and nonnegative integrable functions f, g defined respectively on Ω and Ω^*. Suppose that a point source of light is centered at the origin O and for every $X \in \Omega$ we issue a ray from O passing through X, which after refraction from the unknown surface Γ is another ray given by a unit direction

$Y = Y(X) \in \Omega^*$. It is clear that mapping Y is determined by Γ. Let $f(X)$ be the input intensity of light at $X \in \Omega$ and $g(Y)$ corresponding gain intensity after refraction at $Y \in \Omega^*$. Now the problem can be formulated as follows: given two pairs (Ω, f) and (Ω^*, g) satisfying to energy balance condition

$$\int_{\Omega} f = \int_{\Omega^*} g, \tag{1.1}$$

find a surface Γ, such that for corresponding mapping $Y(X)$ we have

$$Y(\Omega) = \Omega^*.$$

We seek a Γ as a radial graph of a unknown function ρ i.e. $\Gamma = \{Z \in \mathbf{R}^{n+1}, Z = X\rho(X)\}$, then mathematically this problem is amount to solve a Monge-Ampère type equation

$$\det(D^2_{ij}\rho - \sigma_{ij}(x, \rho, D\rho)) = h(x, \rho, D\rho), \tag{1.2}$$

subject to boundary condition

$$Y(\Omega) = \Omega^*. \tag{1.3}$$

Here the derivatives are taken in some orthogonal coordinate system (see Theorem 1) and Ω is a subset of upper half sphere. The solutions to (1.2), should be sought in the class of functions such that the matrix $D^2_{ij}\rho - \sigma_{ij}(x, \rho, D\rho) \geq 0$. It is easy to see that if $\rho \in C^2$ such that $D^2_{ij}\rho - \sigma_{ij} \geq 0$ then equation (1.2) is elliptic with respect to ρ.

It turns out that ρ is a potential function to an optimal transfer problem with a logarithmic cost function [1]

$$c(X, Y) = \begin{cases} \log \frac{1}{\varepsilon(X \cdot Y) - 1}, & \varepsilon > 1, \ X \cdot Y > k, \\ \log \frac{1}{1 - \varepsilon(X \cdot Y)}, & \varepsilon < 1, \ X \cdot Y < k. \end{cases}$$

A similar cost function appears in the reflector problem introduced by X-J. Wang [8], [9]. The regularity of the solutions to optimal transfer problems is discussed in [3] and [5]. The most important thing is the so-called A3 condition, imposed on matrix σ_{ij} [3]. As soon as one has it the rest of the regularity, both local and global will follow from the classical framework established in [3], [5] and [6]. In [1] authors have verified the A3 condition, however without using Euclidian coordinates.

In this note we give a simple way of verifying the A3 condition, for $k < 1$ without invoking to covariant derivatives. It is also explicit, strict and straightforward (3.4). Main idea is to find a simple formula for mapping $Y(X)$ using a parametrization of upper unit half sphere, used in [2]. Then the rest will follow along the arguments of [2]. This method is very general and one can apply it to *near-field* problem. Indeed if one considers a map $z = \rho x + ty$, where t is the stretch function, then $\det Dz$ will give the equation for near-field problem. However we don't discuss this problem in the present note. It is worth noting that, if support functions are ellipsoids, i.e. $k > 1$ the A3 condition is not fulfilled (see (3.4)).

1.1 Notations

Let us consider the case of two homogeneous medias, with refractive constants n_1 and n_2. Ω and Ω^* are two domains on the unit sphere $\mathcal{S}^n = \{X = (x_1, \dots, x_{n+1}), x_1^2 + \dots + x_{n+1}^2 = 1\}$. For $X \in \mathcal{S}^n, x = (x_1, \dots, x_n, 0)$. We also suppose that Ω is a subset of upper unit sphere $\mathcal{S}^n_+ = \mathcal{S}^n \cap \{x_{n+1} > 0\}$. In what follows we consider ρ as a function of $x \in \Omega_0$, with Ω_0 as orthogonal projection of Ω on to hyperplane $x_{n+1} = 0$. By $D\rho$ we denote the gradient of function ρ with respect to x variable $D\rho = (D_{x_1}\rho, \dots, D_{x_n}\rho, 0)$. The reciprocal of ρ is defined as $u = 1/\rho$. We also define two auxiliary functions $b = u^2 + |Du|^2 - (Du \cdot x)^2$ and $V = \sqrt{u^2 - \sigma b} + u$. In what follows $\sigma = (k^2 - 1)/k^2 = 1 - \varepsilon^2$.

1.2 The main results

Our main result is contained in the following

Theorem 1 *If ρ is the radial function defining Γ, and $u = 1/\rho$, then u is a weak solution to*

$$det\left\{\frac{V - \sigma(u - Du \cdot x)}{\sigma}\left(Id + \frac{x \otimes x}{1 - |x|^2}\right) - D^2u\right\} = h, \ if k < 1, \tag{1.4}$$

$$det\left\{D^2u - \frac{V - \sigma(u - Du \cdot x)}{\sigma}(Id + \frac{x \otimes x}{1 - |x|^2})\right\} = h, \ if k > 1,$$

$$h = \frac{f(X)}{g(Y)}k\frac{\sqrt{u^2 - \sigma b}}{u}\frac{1}{(1 - |x|^2)}\left(\frac{V}{k|\sigma|}\right)^n,$$

where $b = u^2 + |Du|^2 - (Du \cdot x)^2, V = \sqrt{u^2 - \sigma b} + u$.

If we set $F = \sigma^{-1}(V - \sigma(u - Du \cdot x))$ and $I = Id + \frac{x \otimes x}{1 - |x|^2}$, the first fundamental form of the upper unit half sphere, then equation can be rewritten as $\det(IF - D^2u) = h$ for $k < 1$. The weak solutions for this equations can be defined through the theory of optimal transfers [1] (see [7] for the discussion of such problems). The higher regularity of the weak solutions depends on the properties of the function F. More precisely we have

Theorem 2 *If $k < 1$ (i.e. when the support functions are hyperboloids of revolution touching Γ from below) and (f, Ω) and (g, Ω^*) satisfy to the regularity assumptions as in [3], [5] and [6] then F is strictly concave as a function of the gradient and the weak solutions are locally (globally) smooth provided f, g are positive smooth functions and $\overline{\Omega}, \overline{\Omega^*} \subset \mathcal{S}^n_+$.*

If $k > 1$ (i.e. when the support functions are ellipsoids of revolution touching Γ from above) then F is not convex in gradient and the weak solutions may not be C^1 even for smooth positive intensities f, g.

2 The main formulas

In this section we derive a simple and useful formula for Y. We use it to compute the Jacobian determinant in the next section.

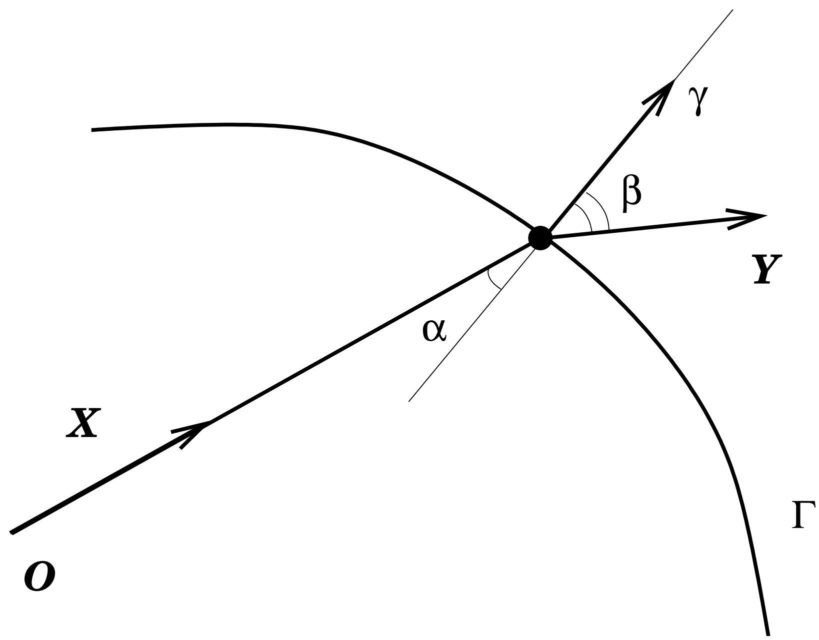

Figure 1: The refraction law.

2.1 The mapping Y

Let Y be the unit direction of the refracted ray. First let us derive a formula for Y, using angles α and β (see figure 1). Since X, Y and outward unit vector γ lie in the same plane, we have

$$Y = C_1 X + C_2 \gamma$$

for two unknowns, C_1 and C_2 depending on X. If one takes the scalar product of Y with γ and then with X, then

$$\begin{cases} \cos\beta = C_1 \cos\alpha + C_2 \\ \cos(\alpha - \beta) = C_1 + C_2 \cos\alpha. \end{cases}$$

Multiplying the first equation by $\cos\alpha$ and subtracting from the second one we infer

$$C_1 = \frac{\sin\beta}{\sin\alpha}, \qquad C_2 = \cos\beta - C_1 \cos\alpha.$$

Introduce $k = n_1/n_2$, hence we find that $C_1 = k$ and $C_2 = \cos\beta - k\cos\alpha$, that is

$$Y = kX + (\cos\beta - k\cos\alpha)\gamma. \tag{2.1}$$

We can further manipulate (2.1). Note that

$$n_2^2 - n_2^2 \cos^2\beta = n_2^2 \sin^2\beta = n_1^2 \sin^2\alpha = n_1^2 - n_1^2 \cos^2\alpha.$$

Dividing the both sides by n_2^2 we obtain

$$k^2 \cos^2\alpha = (k^2 - 1) + \cos^2\beta.$$

Returning to (2.1) we get

$$
\begin{aligned}
Y &= kX + (\sqrt{k^2\cos^2\alpha - (k^2-1)} - k\cos\alpha)\gamma = \\
&= k\left(X + [\sqrt{(X\cdot\gamma)^2 - \sigma} - X\cdot\gamma]\gamma\right),
\end{aligned} \tag{2.2}
$$

where $\sigma = (k^2-1)/k^2$. From [2] we have

$$
\gamma = -\frac{D\rho - X(\rho + D\rho\cdot x)}{\sqrt{\rho^2 + |D\rho|^2 - (D\rho\cdot x)^2}}
$$

where $X = (x, \sqrt{1-|x|^2}), D\rho = (\rho_{x_1}, \dots, \rho_{x_n})$. It is convenient to work with a new function $u = \rho^{-1}$. By direct computation we have that

$$
\gamma = \frac{Du + X(u - Du\cdot x)}{\sqrt{u^2 + |Du|^2 - (Du\cdot x)^2}}.
$$

Introduce $b = u^2 + |Du|^2 - (Du\cdot x)^2$, then

$$
\begin{aligned}
Y &= k\left(X + [\sqrt{(X\cdot\gamma)^2 - \sigma} - X\cdot\gamma]\gamma\right) \\
&= k\left(X + [\sqrt{\frac{u^2}{b} - \sigma} - \frac{u}{\sqrt{b}}]\gamma\right) \\
&= k\left(X + b^{-1}[\sqrt{u^2 - \sigma b} - u][Du + X(u - Du\cdot x)]\right),
\end{aligned}
$$

where we used the fact that

$$
X\cdot\gamma = \frac{u}{\sqrt{u^2 + |Du|^2 - (Du\cdot x)^2}} > 0.
$$

In particular it follows from the previous formula that

$$
Y_{n+1} = kX_{n+1}(1 - \frac{\sigma}{V}(u - (Du\cdot x))). \tag{2.3}
$$

2.2 The Jacobian determinant

Let dX and dY be respectively the area elements corresponding to Ω and Ω^*. Then $dx = X_{n+1}dX$. Recall that Y is a unit vector and denote $y = (Y_1, Y_2, \dots, Y_n, 0) \in \Omega_0^*$, where Ω_0^* is the orthogonal projection of Ω^* onto hyperplane $x_{n+1} = 0$ so we conclude $dy = Y_{n+1}dY$. Hence if we consider y to be a mapping from Ω_0 to Ω_0^* then $dy = |\text{det}Dy|dx$.

For perfect refractor Γ we have the energy balance condition

$$
\int_E f(X)dX = \int_{Y(E)} g(Y)dY, \ \forall \text{ measurable } E \subset \Omega.
$$

Thus we obtain $fdX = gdY$ or

$$
J = \frac{X_{n+1}}{Y_{n+1}}|\text{det}Dy| = \frac{f(X)}{g(Y)} = \frac{dX}{dY}. \tag{2.4}
$$

Thus to find the Jacobian determinant J it is enough to compute $|\det Dy|$.

Before starting our computations let us note, that if $\mu = Id + C\xi \otimes \eta$ for some constant C and for any two vectors $\xi, \eta \in \mathbf{R}^n$, then one has

$$\mu^{-1} = Id - \frac{C\xi \otimes \eta}{1 + C(\xi \cdot \eta)}, \quad \det\mu = 1 + C(\xi \cdot \eta). \tag{2.5}$$

3 Proofs of Theorems 1-2

The main goal of this section is to prove the following

Proposition 1 *If Y is given as above and*

$$y = k\left[x - \frac{\sigma}{V}(Du + x(u - Du \cdot x))\right],$$

then

$$Dy = k\frac{\sigma}{V}\mu[Id - x \otimes x]\left\{(Id + \frac{x \otimes x}{1 - |x|^2})\frac{V - \sigma(u - Du \cdot x)}{\sigma} - D^2 u\right\},$$

where $b = u^2 + |Du|^2 - (Du \cdot x)^2, V = \sqrt{u^2 - \sigma b} + u$ and μ is defined by (3.2).

Proof. Introduce $V = \sqrt{u^2 - \sigma b} + u, z = Du + x(u - Du \cdot x)$. Using these notations one can rewrite

$$y = k[x - \frac{\sigma}{V}z].$$

By a direct computation we have

$$\frac{y_{ij}}{k} = \delta_{ij} - \frac{\delta}{V}(z^i_j - \frac{z^i V_j}{V}).$$

Differentiating z^i and V with respect x_j yields

$$\begin{aligned} z^i_j &= u_{ij} - x_i x_m u_{m,j} + \delta_{ij}(u - Du \cdot x), \\ V_j &= pu_j - q(u_m - (Du \cdot x)x_m)u_{mj}, \end{aligned}$$

where

$$\begin{aligned} p &= \frac{V - \sigma(u - Du \cdot x)}{V - u}, \\ q &= \frac{\sigma}{V - u}. \end{aligned}$$

Then

$$\begin{aligned}\frac{Dy}{k} &= Id - \frac{\sigma}{V}\Big[(Id - x\otimes x)D^2u + Id(u - Du\cdot x) - \frac{p}{V}z\otimes Du \\ &\quad + \frac{q}{V}z\otimes(Du - (Du\cdot x)x)D^2u\Big] \\ &= [1 - \frac{\sigma}{V}(u - Du\cdot x)]\Big[Id + Az\otimes Du \\ &\quad - B\Big\{(Id - x\otimes x) + \frac{q}{V}z\otimes(Du - (Du\cdot x)x)\Big\}D^2u\Big],\end{aligned} \tag{3.1}$$

where we set

$$\begin{aligned}A &= \frac{\frac{\sigma p}{V^2}}{1 - \frac{\sigma}{V}(u - Du\cdot x)} = \frac{\sigma}{V(V-u)}, \\ B &= \frac{\frac{\sigma}{V}}{1 - \frac{\sigma}{V}(u - Du\cdot x)} = \frac{\sigma}{V - \sigma(u - Du\cdot x)}.\end{aligned}$$

Then using Lemma 1 (see below) we finally obtain

$$\begin{aligned}\frac{Dy}{k} &= [1 - \frac{\sigma}{V}(u - Du\cdot x)]B\mu[Id - x\otimes x]\left\{(Id + \frac{x\otimes x}{1-|x|^2})\frac{1}{B} - D^2u\right\} \\ &= \frac{\sigma}{V}\mu[Id - x\otimes x]\left\{(Id + \frac{x\otimes x}{1-|x|^2})\frac{1}{B} - D^2u\right\} \\ &= \frac{\sigma}{V}\mu[Id - x\otimes x]\left\{\frac{V - \sigma(u - Du\cdot x)}{\sigma}(Id + \frac{x\otimes x}{1-|x|^2}) - D^2u\right\}.\end{aligned}$$

Hence to finish the proof of Proposition 1 it remains to prove

Lemma 1 *Let $\mu = Id + Az\otimes Du$, then*

$$\mu^{-1}\left\{(Id - x\otimes x) + \frac{q}{V}z\otimes(Du - (Du\cdot x)x)\right\} = Id - x\otimes x, \tag{3.2}$$

$$det\mu = \frac{Y_{n+1}}{kX_{n+1}}\frac{u}{\sqrt{u^2 - \sigma b}}. \tag{3.3}$$

Proof. First by (2.5)

$$\mu^{-1} = Id - \frac{Az\otimes Du}{1 + A(z\cdot Du)}.$$

Let $\mathcal{N} = \left\{(Id - x\otimes x) + \frac{q}{V}z\otimes(Du - (Du\cdot x)x)\right\}$, then by a direct computation we have

$$\begin{aligned}\mu^{-1}\mathcal{N} &= (Id - x\otimes x) + \frac{q}{V}z\otimes(Du - (Du\cdot x)x) - \frac{Az\otimes Du}{1 + A(z\cdot Du)} \\ &\quad + \frac{A}{1 + A(z\cdot Du)}[(Du\cdot x)z\otimes x - \frac{q}{V}(Du\cdot z)z\otimes(Du - (Du\cdot x)x)].\end{aligned}$$

Let us sum up all $\otimes$ products with z, the resulting vector is

$$\frac{q}{V}(Du-(Du\cdot x)x)+\frac{A}{1+A(z\cdot Du)}\Big\{-Du+(Du\cdot x)x-\frac{q}{V}(Du\cdot z)(Du-(Du\cdot x)x)\Big\}$$
$$=\Big\{\frac{q}{V}-\frac{A}{1+A(z\cdot Du)}(1+\frac{q}{V}Du\cdot z)\Big\}(Du-(Du\cdot x)x).$$

On the other hand

$$\frac{q}{V}-\frac{A}{1+A(z\cdot Du)}(1+\frac{q}{V}Du\cdot z)=\frac{1}{1+A(z\cdot Du)}[\frac{q}{V}-A].$$

Using definitions of q, p and A we obtain that

$$\begin{aligned}\frac{q}{V}-A &= \frac{\sigma}{V(V-u)}-\frac{\sigma p}{V(V-\sigma(u-Du\cdot x))}\\ &= \frac{\sigma}{V}\left\{\frac{1}{V-u}-\frac{\frac{V-\sigma(u-Du\cdot x)}{V-u}}{V-\sigma(u-Du\cdot x)}\right\}\\ &= 0.\end{aligned}$$

To prove (3.3) we notice that $A=\frac{\sigma}{V(V-u)}$. Then using (2.5) and $V=\sqrt{u^2-\sigma b}+u$ we have

$$\begin{aligned}\det\mu &= 1+\frac{\sigma}{V(V-u)}[|Du|^2+u(Du\cdot x)-(Du\cdot x)^2]\\ &= \frac{1}{V(V-u)}[uV-u^2\sigma+\sigma u(Du\cdot x)]\\ &= \frac{u}{V-u}\left\{1-\frac{\sigma}{V}(u-(Du\cdot x))\right\}\end{aligned}$$

and (3.3) follows from (2.3). □

3.1 Ellipsoids and hyperboloids of revolution

In this section we show that $\mathcal{W}=IF-D^2u\equiv 0$ for $u=\frac{1}{C}(1-\varepsilon(\ell\cdot X))$, that is when $\rho=1/u$ is the radial graph of ellipsoid or hyperboloid of revolution. To fix ideas we assume that $\ell=e_{n+1}$. Thus $u=\frac{1}{C}(1-\varepsilon X_{n+1})$. It is enough to show that $B=CX_{n+1}/\varepsilon$. By direct computation

$$b=\frac{1}{C^2}(1-2\varepsilon X_{n+1}+\varepsilon^2)$$
$$u^2-\sigma b=\frac{\varepsilon^2}{C^2}(X_{n+1}-\varepsilon)^2.$$

Therefore $V=(1-\varepsilon^2)/C$, which implies that

$$B=\frac{\sigma}{V-\sigma(u-Du\cdot x)}=\frac{CX_{n+1}}{\varepsilon}.$$

3.2 Proof of Theorem 1

From Proposition 1, Lemma 1 and (2.5) we have that

$$\begin{aligned} \det Dy &= \left(k\frac{\sigma}{V}\right)^n \det\mu \det(Id - x\otimes x)\det\mathcal{W} \\ &= \left(k\frac{\sigma}{V}\right)^n \frac{Y_{n+1}}{kX_{n+1}}\frac{u}{\sqrt{u^2-\sigma b}}(1-|x|^2)\det\mathcal{W} \end{aligned}$$

where $\mathcal{W} = IF - D^2u$ i.e.

$$\mathcal{W} = \frac{V-\sigma(u-Du\cdot x)}{\sigma}(Id + \frac{x\otimes x}{1-|x|^2}) - D^2u.$$

Notice that if Γ is smooth and has support hyperboloids from inside at each point then $\mathcal{W}\geq 0$ and $\mathcal{W}\leq 0$ if Γ has support ellipsoids from outside. Then from (2.4) the Theorem 1 follows.

3.3 Proof of Theorem 2

The equation (1.4) is generalized Monge-Ampère equation. To obtain smoothness of the solution, one needs to show, that $F = \frac{V-\sigma(u-Du\cdot x)}{\sigma}$ is strictly concave in gradient. This is a necessary condition, called A3 and first introduced in [3], in order to obtain C^2 a priori estimates. It turns out that if $\sigma < 0$, i.e. when support functions are hyperboloids of revolution, then F is strictly concave in gradient. Recall that $V = \sqrt{u^2-\sigma b} + u$, hence it is enough to show that $\sqrt{u^2-\sigma b}$ is convex in gradient. Let ξ be the dummy variable for Du, then we have

$$\begin{aligned} \frac{\partial}{\partial\xi_k}\sqrt{u^2-\sigma b} &= -\frac{\sigma}{\sqrt{u^2-\sigma b}}(\xi_k - (\xi\cdot x)x_k), \\ \frac{\partial^2}{\partial\xi_k\partial\xi_l}\sqrt{u^2-\sigma b} &= -\frac{\sigma}{\sqrt{u^2-\sigma b}}\left\{\delta_{lk} - x_kx_l + \sigma\frac{(\xi_k-(\xi\cdot x)x_k)(\xi_l-(\xi\cdot x)x_l)}{u^2-\sigma b}\right\}. \end{aligned}$$

On the other hand $b = u^2 + |\xi|^2 - (\xi\cdot x)^2$, which is strictly convex function of ξ, provided $|x| < 1$. For any $\eta\in\mathbf{R}^n$ we have

$$\begin{aligned} -(u^2-\sigma b)^{\frac{3}{2}}\frac{\partial^2 F}{\partial\xi_k\partial\xi_l}\eta_k\eta_l &= -(u^2-\sigma b)^{\frac{3}{2}}\frac{1}{\sigma}\sum_{k,l}\frac{\partial^2\sqrt{u^2-\sigma b}}{\partial\xi_k\partial\xi_l}\eta_k\eta_l \qquad (3.4) \\ &= (u^2-\sigma b)(|\eta|^2 - (\eta\cdot x)^2) + \sigma[\xi\cdot\eta - (\xi\cdot\eta)(\eta\cdot\xi)]^2 \end{aligned}$$

substituting the value of $b = |\xi|^2 - (\xi\cdot x)^2 + u^2$ we have

$$\begin{aligned} &= u^2(1-\sigma)(|\eta|^2-(\eta\cdot x)^2) + \\ &+\sigma\left\{-|\xi|^2|\eta|^2 + |\xi|^2(\eta\cdot x)^2 + |\eta|^2(\xi\cdot x)^2 + (\xi\cdot\eta)^2 - 2(\xi\cdot\eta)(\xi\cdot x)(\eta\cdot x)\right\}. \end{aligned}$$

The first term is nonnegative since $|x| < 1$ and $(\eta\cdot x)\leq|\eta||x| < |\eta|$. Recall that $\sigma < 0$. Hence it is enough to show that

$$-|\xi|^2|\eta|^2 + |\xi|^2(\eta\cdot x)^2 + |\eta|^2(\xi\cdot x)^2 + (\xi\cdot\eta)^2 - 2(\xi\cdot\eta)(\xi\cdot x)(\eta\cdot x) < 0. \qquad (3.5)$$

This expression is homogeneous in η and ξ thus we may assume that $|\xi| = |\eta| = 1$. Furthermore let x' be the orthogonal projection of x on the two dimensional space spanned by ξ and η. Then (3.5) is equivalent to

$$(\eta \cdot x')^2 + (\xi \cdot x')^2 + (\xi \cdot \eta)^2 - 2(\xi \cdot \eta)(\xi \cdot x')(\eta \cdot x') - 1 < 0.$$

If α, β and γ are the angles between respectively η and x', ξ and x' and η and ξ then $\cos\gamma = \cos(\alpha \pm \beta)$. Thus we have

$$\begin{aligned}|x'|^2(\cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta\cos\gamma) + \cos^2\gamma - 1 < \\ (\cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta\cos\gamma) + \cos^2\gamma - 1 = 0.\end{aligned}$$

From here the proof of Theorem 1 follows from [3] and [6].

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