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On the Divergence of Fourier Series in the General Haar System

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Abstract. For any countable set $D \subset [0,1]$, we construct a bounded measurable function f such that the Fourier series of f with respect to the regular general Haar system is divergent on D and convergent on $[0,1]\backslash D$.

Key Words: Fourier series, general Haar system, divergence sets
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1 Introduction

Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of functions, $f_n\colon [0,1] \to \mathbb{R}$ for all n.

For a functional series $\sum\limits_{n=1}^{\infty} f_n(x)$, the set $D \subset [0,1]$ is called a divergence set if the series is divergent for any $x \in D$ and is convergent when $x \notin D$.

There are many results concerning divergence sets for the Fourier series with respect to classical systems. A. Haar [9] proved that the Fourier-Haar series of any function continuous on $[0,1]$ is uniformly convergent, and for any measurable function, its Fourier-Haar series is convergent almost everywhere on $[0,1]$. V. Prokhorenko [18] proved that for any countable set $F \subset [0,1]$, there exists a bounded function such that the Fourier-Haar series of that function is divergent on F and convergent on $[0,1]\backslash F$. V. Bugadze [2] proved that for any set with 0 measure, there exists a bounded function such that its Fourier-Haar series is divergent on that set.

Other interesting results on divergence sets of the Fourier-Haar series can be found, for example, in [12] and [17]. For similar results for the Fourier-Walsh series see [7], [14], [15], and for the trigonometric Fourier series see [5], [6], [10], [19], [20], [21].

In this paper, we consider the Fourier series with respect to the classical Haar system. Particularly, we prove the following theorem.

Theorem 1 *For any countable set $D \subset [0,1]$, there exists a bounded measurable function $f\colon [0,1] \to \mathbb{R}$ such that D is a divergence set for the Fourier series of f with respect to the regular general Haar system.*

Note that Theorem 1 is a generalization of Prokhorenko's theorem mentioned above. The methods used in the proof of the Theorem 1 are generalizations of the methods used in [18].

The following question remains open: Is Theorem 1 true for every general Haar system (not regular)?

2 Notations

Let us recall the definition of the general Haar system $\{h_n\}_{n=1}^{\infty}$ normalized in $L^2[0,1]$ (see [11], [16]).

For $t_0 = 0$, $t_1 = 1$, let $A_1^{(1)} = [0,1]$ and define $h_1(x)$ by

$$h_1(x) := \chi_{[0,1]}(x),$$

where by χ_E we denote the characteristic function of the set E.

For $t_2 \in (0,1)$, let $A_1^{(2)} = [0,t_2)$, $A_2^{(2)} = [t_2,1]$, $\Delta_2 = A_1^{(1)} = [0,1]$, $\Delta_2^+ = [0,t_2)$, $\Delta_2^- = [t_2,1]$, and put

$$h_2(x) := \begin{cases} \sqrt{\dfrac{\mu\left(\Delta_2^-\right)}{\mu\left(\Delta_2^+\right)\mu\left(\Delta_2\right)}}, & \text{if } x \in \Delta_2^+, \\ -\sqrt{\dfrac{\mu\left(\Delta_2^+\right)}{\mu\left(\Delta_2^-\right)\mu\left(\Delta_2\right)}}, & \text{if } x \in \Delta_2^-, \end{cases}$$

where $\mu(A)$ stands for the Lebesgue measure of the measurable set A.

Suppose now that $t_0, t_1, \ldots, t_n$ $(n \geq 2)$ are already chosen. Let $A_1^{(n)}$, $A_2^{(n)}, \ldots, A_n^{(n)}$ be intervals, enumerated from the left to the right, obtained after splitting $[0,1]$ by $\{t_k\}_{k=2}^{n}$ points. Note that each interval $A_k^{(n)}$, $1 \leq k < n$, is right-open, while $A_n^{(n)}$ is closed. Thus, every point from $[0,1]$ is exactly in one interval $A_k^{(n)}$, $1 \leq k \leq n$.

Let $t_{n+1} \in (0,1)\backslash\{t_2, \ldots, t_n\}$ be the next point, and suppose $t_{n+1} \in A_{k_0}^{(n)}$ for some $k_0 \in [1,n]$. If $k_0 = n$, put $\Delta_{n+1} = A_n^{(n)} = [a,1]$ and let $\Delta_{n+1}^+ = [a, t_{n+1})$, $\Delta_{n+1}^- = [t_{n+1}, 1]$. If $1 \leq k_0 < n$, put $\Delta_{n+1} = A_{k_0}^{(n)} = [b,c)$, $\Delta_{n+1}^+ = [b, t_{n+1})$, $\Delta_{n+1}^- = [t_{n+1}, c)$, and define $h_{n+1}(x)$ by

$$h_{n+1}(x) := \begin{cases} \sqrt{\dfrac{\mu\left(\Delta_{n+1}^-\right)}{\mu\left(\Delta_{n+1}^+\right)\mu\left(\Delta_{n+1}\right)}}, & \text{if } x \in \Delta_{n+1}^+, \\ -\sqrt{\dfrac{\mu\left(\Delta_{n+1}^+\right)}{\mu\left(\Delta_{n+1}^-\right)\mu\left(\Delta_{n+1}\right)}}, & \text{if } x \in \Delta_{n+1}^-, \\ 0, & \text{if } x \in [0,1]\backslash\Delta_{n+1}. \end{cases}$$

The only requirement for the points t_n is that the set $\mathcal{T} = \{t_k\}_{k=0}^{\infty}$ be dense in $[0,1]$, i.e.,

$$\lim_{n\to\infty} \max_{1\le k\le n} \mu\left(A_k^{(n)}\right) = 0. \tag{1}$$

Note that if $\mathcal{T} = \left\{0, 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \dots\right\}$, we get the classical Haar system (see [1, chapter 1, §6], [13, chapter 3, §1]).

For each dense in $[0,1]$ set $\mathcal{T}$, the corresponding Haar system is a complete orthonormal system in $L^2[0,1]$, and it forms a basis in each $L^p[0,1]$, $1 \le p < \infty$. Since the Haar system forms a martingale differences, from Burkholders results on unconditionality of martingale differences ([3], [4]), it follows that every general Haar system is an unconditional basis in $L^p[0,1]$, $1 < p < \infty$.

The general Haar system is called regular (see [8]) if there exists a real number $\lambda \ge 1$ such that for any natural number $n > 1$

$$\frac{1}{\lambda} \le \frac{\mu\left(\Delta_n^+\right)}{\mu\left(\Delta_n^-\right)} \le \lambda. \tag{2}$$

The classical Haar system is regular with $\lambda = 1$.

Note that for any $x \in [0,1]$ and n in $\mathbb{N}$, there exists $k_0 \in [1,n]$ such that $x \in A_{k_0}^{(n)}$. We set

$$A(x,n) := A_{k_0}^{(n)},$$

For a function f, denote by $c_n(f)$ its Fourier coefficients with respect to the general Haar system:

$$c_n(f) := \int_0^1 f(t)h_n(t)dt, \qquad n \ge 1,$$

and put

$$S_n(f;x) := \sum_{k=1}^{n} c_k(f)h_k(x), \qquad x \in [0,1],\ n \ge 1.$$

An important property of the classical Haar system is that the partial sums of the Fourier-Haar series can be expressed by means of integrals on dyadic intervals (see [1, chapter 1, §6], [13, chapter 3, §1]).

It is not difficult to see that the general Haar system has the same property, that is, for each function $f \in L^1[0,1]$ and $n \ge 1$,

$$S_n(f;x) = \frac{1}{\mu\left(A(x,n)\right)} \int_{A(x,n)} f(t)dt \quad \text{for any } x \in [0,1] \tag{3}$$

From (3) it follows that for each continuous function f its Fourier series with respect to the general Haar system converges uniformly, and for each function $f \in L^1[0,1]$, the series converges almost everywhere on $[0,1]$.

3 Auxiliary Lemma

To prove our main result, we need the following auxiliary lemma.

Lemma 1 *Let $\{h_n\}_{n=1}^{\infty}$ be a regular general Haar system. Then for any point $x_0 \in [0,1]$, there exists a function $f\colon [0,1] \to \mathbb{R}$ satisfying the following conditions:*

I. $0 \le f(x) \le 1$ for all $x \in [0,1]$;

II. For each point $x \in [0,1]\backslash\{x_0\}$, there exists $n_0 = n_0(x_0, x)$ such that $S_n(f;x) = S_{n_0}(f;x)$ for all $n > n_0$, $n \in \mathbb{N}$;

III. There exist sequences of natural numbers $p_s = p_s(x_0)$ and $q_s = q_s(x_0)$, $s \in \mathbb{N}$, such that $p_s \nearrow \infty$ and $q_s \nearrow \infty$ as $s \to \infty$, $p_s > q_s$, and $|S_{p_s}(f;x_0) - S_{q_s}(f;x_0)| \ge \dfrac{1}{(\lambda+1)^2}$ for all $s \in \mathbb{N}$.

Proof. Let us define inductively an increasing sequence $\{k_i\}_{i=1}^{\infty}$ of natural numbers such that $x_0 \in \Delta_{k_i}$ for every i (such sequence exists due to (1)). For $i = 1$, we get $k_1 = 2$ since $\Delta_2 = [0,1]$ is the first interval including x_0. We set

$$\Delta_{k_i}[x_0] := \begin{cases} \Delta_{k_i}^{+}, & \text{if } x_0 \in \Delta_{k_i}^{+}, \\ \Delta_{k_i}^{-}, & \text{if } x_0 \in \Delta_{k_i}^{-}, \end{cases} \qquad \widetilde{\Delta}_{k_i}[x_0] := \Delta_{k_i} \backslash \Delta_{k_i}[x_0].$$

Since $x_0 \in \Delta_{k_i}[x_0]$, we have

$$\Delta_{k_i}[x_0] = A(x_0, k_i). \tag{4}$$

Choose k_{i+1} such that $\Delta_{k_{i+1}}$ coincides with $\Delta_{k_i}[x_0]$. Then

$$\Delta_{k_{i+1}} = \Delta_{k_i}[x_0] = \Delta_{k_{i+1}}[x_0] \cup \widetilde{\Delta}_{k_{i+1}}[x_0],$$

$$\mu\left(\Delta_{k_{i+1}}\right) = \mu\left(\Delta_{k_i}[x_0]\right) = \mu\left(\Delta_{k_{i+1}}[x_0]\right) + \mu\left(\widetilde{\Delta}_{k_{i+1}}[x_0]\right). \tag{5}$$

Since $\{h_n\}_{n=1}^{\infty}$ is a regular general Haar system (2), we have

$$\frac{1}{\lambda} \le \frac{\mu\left(\Delta_{k_i}[x_0]\right)}{\mu\left(\widetilde{\Delta}_{k_i}[x_0]\right)} \le \lambda, \qquad \lambda \ge 1. \tag{6}$$

Define a function f by

$$f(x) = \chi_{\bigcup\limits_{s=1}^{\infty} E_s}(x), \qquad x \in [0,1], \tag{7}$$

where $E_s = \widetilde{\Delta}_{k_{2s+1}}[x_0]$. It is clear that f satisfies (I.) (see (7)).

Let $x \in [0,1] \backslash \{x_0\}$ and let n_0 be the smallest natural number for which f is constant on $A(x, n_0)$ (see (7)). It is not difficult to see that f is constant on $A(x, n)$ for all $n > n_0$. Taking (3) into account, we immediately get $S_n(f; x) = S_{n_0}(f; x)$ for all $n > n_0$, $n \in \mathbb{N}$.

To verify the statement (III.), define p_s and q_s as follows:

$$p_s := k_{2s},\ q_s := k_{2s-1},\ s \in \mathbb{N}. \tag{8}$$

From (3)–(8) we get

$$|S_{p_s}(f, x_0) - S_{q_s}(f, x_0)| =$$

$$= \left| \frac{1}{\mu\left(A(x_0, k_{2s})\right)} \int\limits_{A(x_0,k_{2s})} f(t)dt - \frac{1}{\mu\left(A(x_0, k_{2s-1})\right)} \int\limits_{A(x_0,k_{2s-1})} f(t)dt \right| =$$

$$= \left| \frac{1}{\mu\left(\Delta_{k_{2s}}[x_0]\right)} \int\limits_{\Delta_{k_{2s}}[x_0]} f(t)dt - \frac{1}{\mu\left(\Delta_{k_{2s-1}}[x_0]\right)} \int\limits_{\Delta_{k_{2s-1}}[x_0]} f(t)dt \right| =$$

$$= \frac{\mu\left(\widetilde{\Delta}_{k_{2s}}[x_0]\right)}{\mu\left(\Delta_{k_{2s}}[x_0]\right)\mu\left(\Delta_{k_{2s-1}}[x_0]\right)} \int\limits_{\Delta_{k_{2s}}[x_0]} f(t)dt \geq$$

$$\geq \frac{\mu\left(\widetilde{\Delta}_{k_{2s}}[x_0]\right)}{\mu\left(\Delta_{k_{2s}}[x_0]\right)\mu\left(\Delta_{k_{2s-1}}[x_0]\right)} \int\limits_{\widetilde{\Delta}_{k_{2s+1}}[x_0]} f(t)dt =$$

$$= \frac{\mu\left(\widetilde{\Delta}_{k_{2s+1}}[x_0]\right)\mu\left(\widetilde{\Delta}_{k_{2s}}[x_0]\right)}{\left(\mu\left(\Delta_{k_{2s+1}}[x_0]\right) + \mu\left(\widetilde{\Delta}_{k_{2s+1}}[x_0]\right)\right)\left(\mu\left(\Delta_{k_{2s}}[x_0]\right) + \mu\left(\widetilde{\Delta}_{k_{2s}}[x_0]\right)\right)} =$$

$$= \frac{1}{\left(\frac{\mu\left(\Delta_{k_{2s+1}}[x_0]\right)}{\mu\left(\widetilde{\Delta}_{k_{2s+1}}[x_0]\right)} + 1\right)\left(\frac{\mu\left(\Delta_{k_{2s}}[x_0]\right)}{\mu\left(\widetilde{\Delta}_{k_{2s}}[x_0]\right)} + 1\right)} \geq \frac{1}{(\lambda+1)^2}.$$

□

4 Proof of the Theorem

Proof. Let $E = \{x_1, x_2, \ldots, x_k, \ldots\}$. Successively applying Lemma 1 for each point $x_k \in E$, we obtain a sequence of functions $\{f_k(x)\}_{k=1}^{\infty}$ such that the following conditions are satisfied:

1. For all $k \in \mathbb{N}$
$$0 \le f_k(x) \le 1 \text{ for all } x \in [0,1]; \tag{9}$$

2. For all $k \in \mathbb{N}$ and $x \in [0,1]\setminus\{x_k\}$, there exists a natural number $n_k = n_k(x_k, x)$ such that
$$S_n(f_k; x) = S_{n_k}(f_k; x) \text{ for all } n > n_k,\ n \in \mathbb{N}; \tag{10}$$

3. For all $k \in \mathbb{N}$, there exist sequences $N_s^{(k)} = N_s^{(k)}(x_k)$, $M_s^{(k)} = M_s^{(k)}(x_k)$, $s \ge 1$, of natural numbers such that $N_s^{(k)} \nearrow \infty$ and $M_s^{(k)} \nearrow \infty$ as $s \to \infty$, and for all $s \ge 1$, $N_s^{(k)} > M_s^{(k)}$ and
$$\left| S_{N_s^{(k)}}(f_k; x_k) - S_{M_s^{(k)}}(f_k; x_k) \right| \ge \frac{1}{(\lambda+1)^2}. \tag{11}$$

From (9) we get that the series
$$\sum_{k=1}^{\infty} (\lambda+1)^{-4k} f_k(x)$$
is uniformly convergent on $[0,1]$. Setting
$$f(x) = \sum_{k=1}^{\infty} (\lambda+1)^{-4k} f_k(x),$$
we obtain
$$S_n(f; x) = \sum_{k=1}^{\infty} (\lambda+1)^{-4k} S_n(f_k; x). \tag{12}$$

First, let us prove that $S_n(f; x)$ is convergent on $[0,1]\setminus E$. Let $x \in [0,1]\setminus E$. For any $\delta > 0$, take $\nu = \nu(\delta)$ such that
$$\sum_{k=\nu+1}^{\infty} (\lambda+1)^{-4k} < \delta. \tag{13}$$

Let $N_0 := \max\{n_1(x_1, x), n_2(x_2, x), \ldots, n_\nu(x_\nu, x)\}$. Taking into account (10), for all $n > N_0$, we get $S_n(f_k, x) = S_{N_0}(f_k, x)$, $k = 1, 2, \ldots, \nu$. Therefore, for all $N, M > N_0$, we have
$$S_N(f_k; x) - S_M(f_k; x) = 0 \qquad \text{for any } k \in [1, \nu]. \tag{14}$$

Since, according to (3) and (9),
$$0 \le S_n(f_k; x) \le 1 \qquad \text{for all } n, k \in \mathbb{N}, \tag{15}$$

from (12)–(15) for all $N, M > N_0$, we obtain

$$|S_N(f;x) - S_M(f;x)| = \left|\sum_{k=1}^{\infty}(\lambda+1)^{-4k}\left(S_N(f_k;x) - S_M(f_k;x)\right)\right| \leq$$

$$\leq \left|\sum_{k=1}^{\nu}(\lambda+1)^{-4k}\left(S_N(f_k;x) - S_M(f_k;x)\right)\right| +$$

$$+ \sum_{k=\nu+1}^{\infty}(\lambda+1)^{-4k}\left|S_N(f_k;x) - S_M(f_k;x)\right| \leq$$

$$\leq \sum_{k=\nu+1}^{\infty}(\lambda+1)^{-4k} < \delta.$$

Now let us prove that $S_n(f;x)$ is divergent on $E = \{x_1, x_2, \ldots, x_k, \ldots\}$. For any $x = x_{k_0} \in E$ and take a natural number j_0 such that (see (10), (11))

$$N_{j_0}^{(k_0)},\ M_{j_0}^{(k_0)} > \max\{n_1(x_1, x_{k_0}), n_2(x_2, x_{k_0}), \ldots, n_{k_0-1}(x_{k_0-1}, x_{k_0})\},$$

and let $N_0 = \min\{N_{j_0}^{(k_0)},\ M_{j_0}^{(k_0)}\}$. From (10) it follows that $S_n(f_k, x_{k_0}) = S_{N_0}(f_k, x_{k_0})$ for any $k = 1, 2, \ldots, k_0-1$ and $n > N_0$. Therefore, for all $j > j_0$, we have

$$S_{N_j^{(k_0)}}(f_k; x_{k_0}) - S_{M_j^{(k_0)}}(f_k; x_{k_0}) = 0, \qquad k \in [1, k_0).$$

From here and (11), (12), (15) it follows that for all $j > j_0$, we can write

$$\left|S_{N_j^{(k_0)}}(f; x_{k_0}) - S_{M_j^{(k_0)}}(f; x_{k_0})\right| =$$

$$= \left|\sum_{k=1}^{\infty}(\lambda+1)^{-4k}\left(S_{N_j^{(k_0)}}(f_k; x_{k_0}) - S_{M_j^{(k_0)}}(f_k; x_{k_0})\right)\right| \geq$$

$$\geq (\lambda+1)^{-4k_0}\left|S_{N_j^{(k_0)}}(f_{k_0}; x_{k_0}) - S_{M_j^{(k_0)}}(f_{k_0}; x_{k_0})\right| -$$

$$- \sum_{k=k_0+1}^{\infty}(\lambda+1)^{-4k}\left|S_{N_j^{(k_0)}}(f_k; x_{k_0}) - S_{M_j^{(k_0)}}(f_k; x_{k_0})\right| -$$

$$- \sum_{k=1}^{k_0-1}(\lambda+1)^{-4k}\left|S_{N_j^{(k_0)}}(f_k; x_{k_0}) - S_{M_j^{(k_0)}}(f_k; x_{k_0})\right| \geq$$

$$\geq (\lambda+1)^{-4k_0}\frac{1}{(\lambda+1)^2} - \sum_{k=k_0+1}^{\infty}(\lambda+1)^{-4k} =$$

$$= \frac{1}{(\lambda+1)^{4k_0+2}} - \frac{(\lambda+1)^{-4(k_0+1)}}{1-(\lambda+1)^{-4}} =$$

$$= \frac{1}{(\lambda+1)^{4k_0+2}} - \frac{1}{(\lambda+1)^{4k_0}((\lambda+1)^4-1)} >$$

$$> \frac{1}{(\lambda+1)^{4k_0+2}} - \frac{1}{(\lambda+1)^{4k_0+3}} = \frac{\lambda}{(\lambda+1)^{4k_0+3}}.$$

This completes the proof. □

References

[1] G. Alexits, *Konvergenz probleme der orthogonalreihen*, Budapest, VEB Deutscher Verlag der Wissenschafter, Berlin, 1960.

[2] V. M. Bugadze, *Divergence of Fourier-Haar series of bounded functions on sets of measure zero*, Math. Notes **51** (1992), no. 5, pp. 437–441. https://doi.org/10.1007/BF01262173

[3] D. L. Burkholder, *Martingale transforms*, Ann. Math. Statist. **37** (1966), no. 6, pp. 1494–1504. https://doi.org/10.1214/aoms/1177699141

[4] D. L. Burkholder, *Boundary value problems and sharp inequalities for martingale transforms*, Ann. Probab. **12** (1984), no. 3, pp. 647–702. https://doi.org/10.1214/aop/1176993220

[5] V. V. Buzdalin, *On infinitely divergent Fourier trigonometric series of continuous functions*, Matem. Zametki **7** (1970), no. 1, pp. 7–18.

[6] V. V. Buzdalin, *Trigonometric Fourier series of continuous functions divergent on a given set* (in Russian), Mathematics of the USSR-Sbornik **24** (1974), no. 1, pp. 79–102. http://dx.doi.org/10.1070/SM1974v024n01ABEH001906

[7] U. Goginava, *On the divergence of Walsh-Fejér means of bounded functions on sets of measure zero*, Acta Math. Hung. **121** (2008), no. 4, pp. 359–369. https://doi.org/10.1007/s10474-008-7219-2

[8] S. L. Gogyan, *On greedy algorithm in $L^1(0,1)$ by regular Haar system*, J. Contemp. Math. Anal. **46** (2011), no. 1, pp. 21–31. https://doi.org/10.3103/S1068362311010043

[9] A. Haar, *Zur theorie der orthogonalen funktionensysteme*, Math. Ann. **69** (1910), no. 3, pp. 331–371. https://doi.org/10.1007/BF01456326

[10] J-P. Kahane and Y. Katznelson, *Sur les ensembles de divergence des series trigonometriques*, Studia Math. **26** (1966), no. 3, pp. 305–306. https://doi.org/10.4064/sm-26-3-305-306

[11] A. Kamont, *General Haar system and Greedy approximation*. Studia Math. **145** (2001), no. 2, pp. 165–184. http://dx.doi.org/10.4064/sm145-2-5

[12] G. A. Karagulyan, *On the complete characterization of divergence sets of Fourier-Haar series*. J. Contemp. Math. Anal. **45** (2010), no.6, pp. 334–347. https://doi.org/10.3103/S1068362310060051

[13] B. S. Kashin and A. A. Sahakyan, *Orthogonal series* (in Russian), Nauka, Moscow, 1984.

[14] Sh. V. Kheladze, *On everywhere divergence of Fourier series in Vilenkin bounded systems* (in Russian), Trudi Tbil. Mat. Inst. AN Gruz. SSR **58** (1978), pp. 225–242.

[15] Sh. V. Kheladze, *On everywhere divergence of Fourier-Walsh series* (in Russian), Soobshch. AN Gruz. SSR **77** (1975), no 2., pp. 305–307.

[16] A. Kh. Kobelyan, *On a property of general Haar system*, Proceedings of the YSU, Physical and Mathematical Sciences **3** (2013), pp. 23–28.

[17] M. A. Lunina, *The set of points of unbounded divergence of series in the Haar system* (in Russian), Vestnik Moskov. Univ. Ser. I Mat. Meh. **31** (1976), no. 4, pp. 13–20.

[18] V. I. Prokhorenko, *Divergent Fourier series with respect to Haar's system* (in Russian), Izv. Vyssh. Uchebn. Zaved. Mat. (1971) no. 1, pp. 62–68.

[19] B. S. Stechkin, *On convergence and divergence of trigonometric series* (in Russian), Uspekhi Mat. Nauk **6** (1951), no. 2 (42), pp. 148–149.

[20] L. V. Taikov, *On the divergence of Fourier series in a re-arranged trigonometric system* (in Russian), Uspekhi Mat. Nauk **18** (1963), no. 5 (113), pp. 191–198.

[21] K. Zeller, *Über konvergenzmengen von Fourierreihen*, Arch. Math. **6** (1955), no. 4, pp. 335–340. https://doi.org/10.1007/BF01899414

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