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Strong Convergence Algorithm for the Split Problem of Variational Inclusions, Split Generalized Equilibrium Problem and Fixed Point Problem

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Abstract. The purpose of this paper is to recommend an iterative scheme to approximate a common element of the solution sets of the split problem of variational inclusions, split generalized equilibrium problem and fixed point problem for non-expansive mappings. We prove that the sequences generated by the recommended iterative scheme strongly converge to a common element of solution sets of stated split problems. In the end, we provide a numerical example to support and justify our main result. The result studied in this paper generalizes and extends some widely recognized results in this direction.

Key Words: Non-expansive mapping, split feasibility problem, averaged mapping, split variational inclusion problem, split generalized equilibrium problem, fixed point problem
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Introduction

We start with introducing some necessary notations. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces equipped with inner product $\langle \cdot\,, \cdot \rangle$ and induced norm $\| \cdot \|$. Let $C \subseteq \mathcal{H}_1$ and $Q \subseteq \mathcal{H}_2$ be two non-empty subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, which are closed and convex.

A mapping $f : \mathcal{H}_1 \to \mathcal{H}_1$ is called τ-Lipschitzian if there exists a constant $\tau \geq 0$ such that

$$\|f(x) - f(y)\| \leq \tau \, \|x - y\|,$$

for any $x, y \in \mathcal{H}_1$. A mapping $U : \mathcal{H}_1 \to \mathcal{H}_1$ is said to be non-expansive if

$$\|U(x) - U(y)\| \leq \|x - y\| \quad \text{for any } x, y \in \mathcal{H}_1.$$

The solution set of U is defined as $\text{Fix}(U) = \{x \in \mathcal{H}_1 : U(x) = x\}$. A mapping $A : \mathcal{H}_1 \to \mathcal{H}_1$ is called strongly positive mapping if there exists a constant $\zeta > 0$ such that

$$< Ax, x > \leq \zeta \|x\|^2, \qquad x \in \mathcal{H}_1,$$

In 1994, Censor and Elfving [8] proposed split feasibility problem (SFP) in the space of finite dimension, which could be characterized as:

$$x \in C \quad \text{and} \quad Ax \in Q. \tag{1}$$

Due to its vast applications in science and engineering, such as signal processing, image reconstruction, medical specialities engineering, geophysics, etc., much attention is paid to this problem (see, for example, [8, 5, 7]). Also, problem (1) is strongly related to some general problems like the convex feasibility problem [4], the multiple-set split feasibility problem (SPF) [9], the split feasibility problem (SPF) [25, 20], and the split common fixed point problem [32].

Now, let $\gamma = \{x \in C : Ax \in Q\}$. For finding a solution to problem (1), Byrne [5, 6] proposed CQ algorithm, which is formulated as follows: For any $x_0 \in \mathcal{H}_1$ define

$$x_{m+1} = P_C(x_m - \eta A^*(I - P_Q)Ax_m), \qquad m \geq 0,$$

where P_C and P_Q denote the projection operators on C and Q, respectively. Afterwards, several different techniques were proposed to solve the problem (1) (see, for example, [29, 31] and the references therein). In 2013, Zhu et.al. [34] proposed the following problem:

$$\text{Find} \qquad x \in C \cap \text{Fix}(U) \qquad \text{such that} \quad Ax \in Q \cap \text{Fix}(V), \tag{2}$$

where U and V are non-expansive mappings on C and Q, respectively. Under some appropriate conditions, the sequence generated by technique proposed in [34] converges strongly to an element of the solution set of the problem (2).

Let $E_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $E_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ be two multi-valued mappings with non-void values, and let f and g be two mappings such that $f : \mathcal{H}_1 \to \mathcal{H}_1$ and $g : \mathcal{H}_2 \to \mathcal{H}_2$. Inspired by the work of Censor et.al. [10], Moudafi [21] proposed the following split monotone variational inclusion problem (SMVIP):

$$\text{Find} \qquad x \in \mathcal{H}_1 \qquad \text{such that} \quad 0 \in f(x) + E_1(x), \tag{3}$$

$$\text{and} \quad y = Ax \in \mathcal{H}_2 \qquad \text{solves} \qquad 0 \in g(x) + E_2(y). \tag{4}$$

Under some acceptable conditions, Moudafi proved that the sequence generated by his algorithm converges weakly to a solution of the problem (2).

Inspired by the work of Moudafi [21], Ansari and Rehan [3] considered the following problem:

$$\begin{aligned} &\text{Find} \qquad x \in \mathrm{Fix}(U) \qquad \text{such that} \qquad 0 \in f(x) + E_1(x), \\ &\text{and} \quad y = Ax \in \mathrm{Fix}(V) \qquad \text{solves} \qquad 0 \in g(x) + E_2(y), \end{aligned} \tag{5}$$

under some suitable conditions, proved the weak convergence of their algorithm.

If $f = 0$ and $g = 0$, the problem (3)–(4) reduces to the following split variational inclusion problem (SVIP):

$$\begin{aligned} &\text{Find} \qquad x \in \mathcal{H}_1 \qquad \text{such that} \qquad 0 \in E_1(x), \\ &\text{and} \quad y = Ax \in \mathcal{H}_2 \qquad \text{solves} \qquad 0 \in E_2(y). \end{aligned} \tag{6}$$

Let $\mathrm{SOL}(E_1) = \{x \in \mathcal{H}_1 : 0 \in E_1(x)\}$ and $\mathrm{SOL}(E_2) = \{x \in \mathcal{H}_2 : 0 \in E_2(x)\}$. In the past few years, many authors have studied and found solutions for SVIP (see, for example, [1], [2]).

In 2018, Majee et.al. [18] considered the following SVIP and fixed point problems: Find $x \in \cap_{i=1}^{M_1} \mathrm{Fix}(U_i) \cap \mathrm{SOL}(E_1)$ such that

$$Ax \in \cap_{i=1}^{M_2} \mathrm{Fix}(V_i) \cap \mathrm{SOL}(E_2), \tag{7}$$

where $U_i : \mathcal{H}_1 \to \mathcal{H}_1$, $i = 1, 2, ..., M_1$, and $V_j : \mathcal{H}_2 \to \mathcal{H}_2$, $j = 1, 2, ..., M_2$, are non-expansive mappings. They define a sequence $\{x_m\}$ as follows: For $x_1 \in \mathcal{H}_1$ and

$$v_m = U_{M_1}^m U_{M_1-1}^m ... U_1^m J_{\lambda,E_1}(x_m + \eta A^*(V_{M_2}^m V_{M_2-1}^m ... V_1^m J_{\lambda,E_2} - I)Ax_m),$$

put

$$x_{m+1} = \sigma_m \gamma f(x_m) + b_m x_m + ((1 - b_m) - \sigma_m \rho W)v_m, \qquad m \geq 1.$$

Also, they establish the strong convergence of their scheme under some appropriate conditions.

In equilibrium problem (EP) for a bi-function $G : C \times C \to \mathbb{R}$, one needs to find $\bar{p} \in C$ such that

$$G(\bar{p}, y) \ \geq \ 0 \qquad \text{for all } y \in C. \tag{8}$$

The solution set for the problem (8) is denoted by $\mathrm{EP}(G)$. Takahashi and Takahashi [26] proposed an iterative method to find a common element of the solution set of the problem (8) and the solution set of fixed point problem of a non-expansive mapping in a Hilbert space. For $x_1 \in \mathcal{H}_1$, they defined the sequences $\{x_m\}$ and $\{w_m\}$ recursively by

$$G_1(w_m, y) + \frac{1}{r_m}\langle y - w_m, w_m - x_m\rangle \geq 0, \qquad y \in C,$$

$$x_{m+1} = \sigma_m f(x_m) + (1-\sigma_m)Tx_m, \qquad m \in \mathbb{N}.$$

Obtaining motivation from the work of Marino and Xu [27] and Takahashi and Takahashi [26], Plublieng and Punpaeng [23] proposed a general iterative method to find a common element of solution set for EP (8) and a fixed point problem for a non-expansive mapping in a Hilbert space. For $x_1 \in \mathcal{H}_1$, they defined the sequences $\{x_m\}$ and $\{w_m\}$ recursively by

$$G_1(w_m, y) + \frac{1}{r_m}\langle y - w_m, w_m - x_m\rangle \geq 0, \qquad y \in C,$$

$$x_{m+1} = \sigma_m \eta f(x_m) + (1-\sigma_m A)Ux_m, \qquad m \in \mathbb{N}.$$

Under some acceptable conditions on sequences $\{\sigma_m\}$ and $\{r_m\}$, they proved that the sequence generated by their algorithm converges strongly to the unique solution of the variational inequality problem:

$$\langle (A-\eta f)z, x-z\rangle \ \geq \ 0, \qquad x \in \mathrm{EP}(G) \cap \mathrm{Fix}(U).$$

To find a common element of the solution set for EP (8) and a fixed point problem for a finite family of μ-strictly pseudocontractive mappings, Peng et.al. [22] introduced the following iterative scheme: For $x_1 \in \mathcal{H}_1$, they defined the sequences $\{x_m\}$, $\{y_m\}$ and $\{w_m\}$ recursively by

$$G(w_m, y) + \phi_(y) - \phi_(w_m) + \frac{1}{r_m}\langle y - w_m, w_m - x_m\rangle \geq 0, \qquad y \in C,$$

$$y_m = \eta_m w_m + (1-\eta_m)\textstyle\sum_{i=1}^{M} \xi_i^m U_i w_m,$$

$$x_{m+1} = \sigma_m f(x_m) + b_m x_m + (1-\sigma_m - b_m)y_m, \qquad m \in \mathbb{N}.$$

Moudafi [21] introduced the following split equilibrium problem (SEP):

$$\text{Find } p^* \in C \qquad \text{such that} \qquad G_1(p^*, x) \geq 0, \quad x \in C, \tag{9}$$

$$\text{and} \quad y^* = Ap^* \in Q \quad \text{satisfies} \quad G_2(y^*, y) \geq 0 \quad \text{for any } y \in Q, \tag{10}$$

where G_1 and G_2 are bi-functions and A is a bounded linear operator. The solution set of the split equilibrium problem (9)–(10) is denoted by $\mathcal{S} = \{p^* \in \mathrm{EP}(G_1) : Ap^* \in \mathrm{EP}(G_2)\}$.

In 2017, Majee et.al. [17] considered the following split generalized equilibrium problem (SGEP):

$$\text{Find } p^* \in C \qquad \text{such that} \qquad G_1(p^*, x) + \phi_1(p^*, x) \geq 0, \quad x \in C, \tag{11}$$

$$\text{and } y^* = Ap^* \in Q \text{ satisfies } G_2(y^*, y) + \phi_2(y^*, y) \geq 0 \text{ for any } y \in Q, \tag{12}$$

where G_1, ϕ_1 and G_2, ϕ_2 are non linear bi-functions, and A is a bounded linear operator. They denote by $\mathrm{GEP}(G_1, \phi_1)$ the solution set for GEP (11) and by $\mathrm{GEP}(G_2, \phi_2)$ the solution set for GEP (12), while the solution set for SGEP (11)–(12) is denoted by $\mathcal{S} = \{p \in \mathrm{EP}(G_1, \phi_1) : Ap \in \mathrm{EP}(G_2, \phi_2)\}$. Further, they define a sequence $\{x_m\}$ as follows: For $x_1 \in \mathcal{H}_1$ and

$$\begin{aligned} w_m &= T_{r_m}^{(G_1,\phi_1)}(x_m + \eta A^*(T_{r_m}^{(G_2,\phi_2)} - I)Ax_m), \\ y_m &= \zeta_m x_m + (1-\zeta_m)V_M^m V_{M-1}^m ... V_1^m u_m, \end{aligned}$$

put

$$x_{m+1} = \sigma_m \gamma f(x_m) + b_m x_m + ((1-b_m) - \sigma_m \rho W)v_m, \qquad m \in \mathbb{N}.$$

Authors of [17] established the strong convergence of their scheme under some appropriate conditions.

Motivated and inspired by the above mentioned works, we suggest and study an iterative scheme to approximate a common element for the solution sets of SVIP (7), SGEP (11)–(12) and a fixed point problem in a real Hilbert space. Further, we provide a numerical example to support and justify our work. Also, we prove strong convergence of the iterative method we used, which is prudent than weak convergence.

The paper is organized in the following manner. In the second section, we recall some definitions and auxiliary results. In the third section, we present our scheme and study its convergence. In the last section, we justify our algorithm with a numerical example.

1 Preliminaries

In this section, we provide definitions, assumptions and lemmas which will be used to prove our main result.

Throughout the paper, we use the symbol $\rightharpoonup$ for weak convergence and the symbol $\to$ for strong convergence.

The mapping $U : \mathcal{H}_1 \to \mathcal{H}_1$ is said to be
(i) monotone if for any $x, y \in \mathcal{H}_1$,

$$\langle Ux - Uy, x - y\rangle \geq 0;$$

(ii) σ-strongly monotone if there exists $\sigma > 0$ such that

$$\langle Ux - Uy, x - y\rangle \geq \sigma\|x - y\|^2 \quad \text{for any } x, y \in \mathcal{H}_1;$$

(iii) firmly non-expansive if for any $x, y \in \mathcal{H}_1$,

$$\langle Ux - Uy, x - y\rangle \geq \|Ux - Uy\|^2,$$

or, equivalently,

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 - \|(I - U)x - (I - U)y\|^2;$$

(iv) μ-strictly pseudo-contraction if there exists a constant $\mu \in [0,1)$ such that

$$\|Ux - Uy\|^2 \leq \|x-y\|^2 + \mu\|(I-U)x - (I-U)y\|^2 \quad \text{for any } x, y \in \mathcal{H}_1.$$

It is well known that every firmly non-expansive mapping is a non-expansive mapping as well. Note also that every non-expansive mapping $U : \mathcal{H}_1 \to \mathcal{H}_1$ satisfies the inequality

$$\langle (x-Ux)-(y-Uy), Uy-Ux\rangle \leq \frac{1}{2}\|(Ux-x)-(Uy-y)\|^2, \quad (x,y) \in \mathcal{H}_1 \times \mathcal{H}_2.$$

Therefore, for all $(x, y) \in \mathcal{H}_1 \times \mathrm{Fix}\,(U)$, we get

$$\langle x - Ux, y - Ux\rangle \leq \frac{1}{2}\|(Ux-x)\|^2. \tag{13}$$

A mapping $f : \mathcal{H}_1 \to \mathcal{H}_1$ is called a contraction if there exists $\tau \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \tau\|x-y\|, \qquad x, y \in \mathcal{H}_1.$$

A mapping $P_C : \mathcal{H}_1 \to C$ is called a metric projection if for each point $x \in \mathcal{H}_1$, there exists a unique nearest point $P_C(x)$ in C such that

$$\|x - P_C(x)\| \leq \|x-y\|, \qquad y \in C.$$

Note that P_C is non-expansive and firmly non-expansive. Moreover, P_C is characterized by the following property:

$$\langle x - P_C(x), y - P_C(x)\rangle \leq 0, \qquad x \in \mathcal{H}_1,\ y \in C. \tag{14}$$

In the Hilbert space $\mathcal{H}_1$, the following inequalities hold for any $x, y \in H_1$:

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \tag{15}$$

$$\|\sigma x + (1-\sigma)y\|^2 = \sigma\|x\|^2 + (1-\sigma)\|y\|^2 - \sigma(1-\sigma)\|x-y\|^2. \tag{16}$$

A multi-valued mapping $E : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ is said to be monotone if for each $x, y \in \mathcal{H}_1$ and any $u \in Ex$, $v \in Ey$,

$$\langle u - v, x - y\rangle \geq 0.$$

For a multi-valued mapping E, graph $\mathcal{G}(E)$ is defined as

$$\mathcal{G}(\mathrm{E}) = \{(\mathrm{x}, \mathrm{u}) \in \mathcal{H}_1 \times \mathcal{H}_1 : \mathrm{u} \in \mathrm{E}(\mathrm{x})\}.$$

A monotone mapping $E : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ is called maximal monotone if the graph of any other monotone mapping does not contain $\mathcal{G}(E)$ properly.

Remark 1 *A monotone mapping is said to be maximal monotone if and only if for* $(x, u) \in \mathcal{H}_1 \times \mathcal{H}_1$, $\langle x - y, u - v\rangle \geq 0$ *for all pairs* $(y, v) \in \mathcal{G}(E)$ *implies that* $u \in Ex$.

Let $E_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ be a multi-valued maximal monotone mapping. For some $\lambda > 0$, the resolvent mapping $J_{\lambda,E_1}(x) : \mathcal{H}_1 \to \mathcal{H}_1$ equipped with E, is defined by

$$J_{\lambda,E_1}(x) = (I + \lambda E_1)^{-1}(x), \qquad x \in \mathcal{H}_1,$$

where I stands for identity operator on $\mathcal{H}_1$.

It is obvious that for all $\lambda > 0$, the resolvent operator J_{λ,E_1} is single-valued, non-expansive and firmly non-expansive. Also, it is known that p^* solves the VIP (6) iff $p^* = J_{\lambda,E_1}(p^*)$.

A mapping $U : \mathcal{H}_1 \to \mathcal{H}_1$ is called averaged mapping if there exists $\sigma \in (0, 1)$ such that $U = (1 - \sigma)I + \sigma V$, where $V : \mathcal{H}_1 \to \mathcal{H}_1$ is a non-expansive mapping. It is well known that an averaged mapping is also a non-expansive mapping and $\mathrm{Fix}(V) = \mathrm{Fix}(U)$ (see [30]).

Lemma 1 [6, 12] *If* $\{U_i\}_{i=1}^M$ *are the averaged mappings with a common fixed point, then*

$$\cap_{i=1}^M \mathrm{Fix}(U_i) = \mathrm{Fix}(U_1U_2...U_M).$$

Particularly, for $M = 2$, $\mathrm{Fix}(U_1) \cap \mathrm{Fix}(U_2) = \mathrm{Fix}(U_1U_2) = \mathrm{Fix}(U_2U_1)$.

Lemma 2 [16] *Let* C *be a non-empty, convex and closed subset of a real Hilbert space* $\mathcal{H}_1$ *and let* $U : C \to C$ *be a non-expansive mapping. If* $\{x_m\}$ *is a sequence in* C *converging weakly to* $x \in C$ *and* $\{(I - U)x_m\}$ *strongly converges to* $y \in C$, *then* $(I - U)x = y$. *In particular, if* $y = 0$, *then* $x \in \mathrm{Fix}(U)$.

The above lemma is called demiclosedness principle.

Lemma 3 [27] *Suppose that* A *is a strongly positive bounded linear operator on a Hilbert space* $\mathcal{H}_1$ *with coefficient* $\bar{\eta} > 0$ *and* $0 < \tau < \|A\|^{-1}$. *Then* $\|I - \tau A\| \leq 1 - \tau\bar{\eta}$.

Lemma 4 [33] *Suppose that* $V : \mathcal{H}_1 \to \mathcal{H}_1$ *is a* μ*-strictly pseudo contractive mapping on a Hilbert space* $\mathcal{H}_1$, *and the mapping* U *is defined by* $Ux = \sigma x + (1 - \sigma)Vx$ *for each* $x \in \mathcal{H}_1$ *where* $\sigma \in [\mu, 1)$. *Then* U *is non-expansive mapping with* $\mathrm{Fix}(U) = \mathrm{Fix}(V)$.

Lemma 5 [13] *Suppose* $\mathcal{H}_1$ *is a Hilbert space. Let* $f : C \to C$ *be a* τ*-Lipschitzian mapping and* $A : \mathcal{H}_1 \to \mathcal{H}_1$ *be a strongly positive bounded linear operator with coefficient* $\zeta > 0$. *If* $\rho\zeta > \gamma\tau$, *then*

$$\langle(\rho A - \gamma f)x - (\rho A - \gamma f)y, x - y\rangle \geq (\rho\zeta - \gamma\tau)\|x - y\|^2, \qquad x, y \in \mathcal{H}_1.$$

That is, $\rho A - \gamma f$ *is strongly monotone with coefficient* $\rho\zeta - \gamma\tau$.

Further, we present some assumptions, which for the first time were considered in [11].
Assumptions 1 Let $G : C \times C \to \mathbb{R}$ be a non linear bi-function satisfying the following conditions: for all $x, y \in C$
(i) $G(x, x) \geq 0$;
(ii) G is monotone, i.e., $G(x, y) + G(y, x) \leq 0$;
(iii) G is upper semi continuous, i.e.,

$$\limsup_{t \to 0} G(tz + (1-t)x, y) \leq G(x, y);$$

(iv) the function $y \mapsto G(x, y)$ is convex and lower semi continuous.
Let $\phi : C \times C \to \mathbb{R}$ be such a function that for all $x, y \in C$,
(a) $\phi(x, x) \leq 0$;
(b) the function $x \mapsto \phi(x, y)$ is upper semi continuous;
(c) the function $y \mapsto \phi(x, y)$ is convex and lower semi continuous.
(d) for a fixed $r > 0$ and $z \in C$, there exists a non-empty, closed, convex and bounded subset $\mathcal{K}$ of $\mathcal{H}_1$ and $x \in C \cap \mathcal{K}$ such that

$$G(y, x) + \phi(y, x) + \frac{1}{r}\langle y - x, x - z\rangle \leq 0, \qquad y \in C \setminus \mathcal{K}.$$

Under these assumptions, the following statements hold.

Lemma 6 [11] *Suppose that the non-linear bi-functions $G_1, \phi_1 : C \times C \to \mathbb{R}$ satisfy Assumptions 1, and $r > 0$, $x \in \mathcal{H}_1$. Then there exists $z \in C$ such that*

$$G_1(z, y) + \phi_1(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \qquad y \in C.$$

Lemma 7 [8] *Suppose that the non-linear bi-functions $G_1, \phi_1 : C \times C \to \mathbb{R}$ satisfy Assumptions 1. For $r > 0$ and for each $x \in \mathcal{H}_1$, define a mapping $T_r^{(G_1,\phi_1)} : \mathcal{H}_1 \to C$ as follows:*

$$T_r^{(G_1,\phi_1)}(x) = \left\{ z \in C : G_1(z, y) + \phi_1(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, y \in C \right\},$$

Then

(i) $T_r^{(G_1,\phi_1)}$ *is non-empty;*

(ii) $T_r^{(G_1,\phi_1)}$ *is firmly non-expansive, i.e.,*

$$\|T_r^{(G_1,\phi_1)}(x) - T_r^{(G_1,\phi_1)}(y)\|^2 \leq \langle T_r^{(G_1,\phi_1)}(x) - T_r^{(G_1,\phi_1)}(y), x - y\rangle, \quad x, y \in \mathcal{H}_1;$$

(iii) $\mathrm{Fix}(T_r^{(G_1,\phi_1)}) = \mathrm{GEP}(G_1, \phi_1)$;

(iv) $T_r^{(G_1,\phi_1)}$ *is single-valued;*

(v) $\mathrm{GEP}(G_1, \phi_1)$ *is convex and closed.*

Suppose that $G_2, \phi_2 : Q \times Q \to \mathbb{R}$ satisfy Assumptions 1. For $s > 0$ and for each $w \in \mathcal{H}_2$, define a mapping $T_s^{(G_2,\phi_2)} : \mathcal{H}_2 \to Q$ as follows:

$$T_s^{(G_2,\phi_2)}(w) = \left\{ c \in Q : G_2(c,e) + \phi_2(c,e) + \frac{1}{s}\langle e - c, c - w\rangle \geq 0, e \in Q \right\}.$$

Then, we observe that $T_s^{(G_2,\phi_2)}$ is non-empty single-valued and firmly non-expansive. Also, $\mathrm{GEP}(G_2, \phi_2, Q)$ is convex and closed, and $\mathrm{Fix}(T_s^{(G_2,\phi_2)}) = \mathrm{GEP}(G_2, \phi_2, Q)$ where $\mathrm{GEP}(G_2, \phi_2, Q)$ is a solution of GEP (11).

We find that $\mathrm{GEP}(G_2, \phi_2) \subseteq \mathrm{GEP}(G_2, \phi_2, Q)$. Furthermore, one can easily prove that $\mathcal{S}$ is convex and closed.

Lemma 8 [19] *Let $G_1 : C \times C \to \mathbb{R}$ be a non linear bi-function satisfying Assumptions 1 and let $T_r^{G_1}$ be defined as in Lemma 7 for $r > 0$. Then for all $x, y \in \mathcal{H}_1$ and $r_1, r_2 > 0$,*

$$\|T_{r_2}^{G_1}(y) - T_{r_1}^{G_1}(y)\| \leq \|y - x\| + \left|\frac{r_2 - r_1}{r_2}\right| \|T_{r_2}^{G_1}(y) - y\|.$$

Lemma 9 [28] *Let σ_m be a sequence of non-negative real numbers such that*

$$\sigma_{m+1} \leq (1 - \eta_m)\sigma_m + \zeta_m,$$

where $\{\eta_m\}$ and $\{\zeta_m\}$ are the sequences of real numbers which satisfy the following conditions:

(i) $\eta_m \in (0,1)$ and $\sum_{m=1}^{\infty} \eta_m = \infty$;

(ii) $\limsup_{m\to\infty} \frac{\zeta_m}{\eta_m} \leq 0$ or $\sum_{m=1}^{\infty} |\zeta_m| < \infty$.

Then $\lim_{m\to\infty} \sigma_m = 0$.

Lemma 10 [24] *Let $\{x_m\}$ and $\{z_m\}$ be two bounded sequences in a Banach space X and let $\{b_m\}$ be a sequence in $[0,1]$ which satisfies the following condition:*

$$0 < \liminf_{m\to\infty} b_m \leq \limsup_{m\to\infty} b_m < 1.$$

Suppose $x_{m+1} = (1 - b_m)z_m + b_m x_m$ for all integers $m \geq 0$, and

$$\limsup_{m\to\infty}(\|z_{m+1} - z_m\| - \|x_{m+1} - x_m\|) \leq 0.$$

Then

$$\lim_{m\to\infty} \|x_m - z_m\| = 0.$$

2 Main Result

In this section, we define our algorithm and provide its convergence analysis.

Suppose $\mathcal{H}_1$ and $\mathcal{H}_2$ are two real Hilbert spaces. Let C and Q be two non-empty closed and convex subset of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let $G_1, \phi_1 : C \times C \to \mathbb{R}$ and $G_2, \phi_2 : Q \times Q \to \mathbb{R}$ be non-linear bi-functions satisfying Assumptions 1, ϕ_1, ϕ_2 be monotone, and G_2 be upper semi continuous in the first argument.

Assumptions 2

(A1) The solution set $\mathcal{F} = \Omega \cap \mathcal{S} \neq \emptyset$, where $\Omega = \{x : x \in \cap_{i=1}^{M_1} \text{Fix}(U_i) \cap \text{SOL}(E_1)$ and $Ax \in \cap_{i=1}^{M_2} \text{Fix}(V_i) \cap \text{SOL}(E_2)\}$ and $\mathcal{S} = \{p \in \text{GEP}(G_1, \phi_1) : Ap \in \text{GEP}(G_2, \phi_2)\}$;

(A2) $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator and $A^* : \mathcal{H}_2 \to \mathcal{H}_1$ is the adjoint operator of A;

(A3) $E_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $E_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ are two maximal monotone operators;

(A4) $\{U_i\}_{i=1}^{M_1}$ and $\{V_i\}_{i=1}^{M_2}$ are two finite families of non-expansive mappings on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively;

(A5) $W : \mathcal{H}_1 \to \mathcal{H}_1$ is a strongly positive bounded linear operator with coefficient $\zeta > 0$;

(A6) $f : \mathcal{H}_1 \to \mathcal{H}_1$ is a τ-Lipschitzian mapping with the coefficient $\tau \geqslant 0$;

(B1) Let $\gamma, \rho > 0$ be such that $\rho\zeta > \gamma\tau$. Let $\{\sigma_m\} \subset (0,1)$ be such that $\lim\limits_{m\to\infty} \sigma_m = 0$, $\sum_{m=0}^{\infty} \sigma_m = 0$, $0 < \sigma_m \leq \min\{1, (\rho\|W\|)^{-1}\}$ and $\{b_m\} \subset (0,1)$ be such that $0 \leq b_m \leq b < 1$ and $0 < \liminf\limits_{m\to\infty} b_m \leq \limsup\limits_{m\to\infty} b_m < 1$ for some $b \in (0,1)$;

(B2) $\{r_m\} \subset (0,\infty)$ is such that $\liminf\limits_{m\to\infty} r_m > 0$ and $\lim\limits_{m\to\infty} \mid r_{m+1} - r_m \mid = 0$;

(B3) $\eta_m^i \in (0,1)$, $\xi_m^i \in (0,1)$, $\lim\limits_{m\to\infty} \mid \eta_{m+1}^i - \eta_m^i \mid = 0$ for $i = 1, 2, ..., M_1$ and $\lim\limits_{m\to\infty} \mid \xi_{m+1}^j - \xi_m^j \mid = 0$ for $j = 1, 2, ..., M_2$.

Supposing Assumptions (A1-B3) are satisfied, we suggest the following algorithm to approximate a common element for the solution sets of SVIP (7), SGEP (11)-(12) and a fixed point problem in a real Hilbert space.

Algorithm For $\lambda > 0$, select $x_1 \in \mathcal{H}_1$, the parameters γ, ρ and the sequences $\{\sigma_m\}$, $\{b_m\}$, $\{r_m\}$ such that Assumptions (B1-B2) are satisfied.

Step 1. Set

$$v_m = U^m_{M_1} U^m_{M_1-1} ... U^m_1 J_{\lambda,E_1}(x_m + \eta A^*(V^m_{M_2} V^m_{M_2-1} ... V^m_1 J_{\lambda,E_2} - I)Ax_m),$$

where $U^i_m = (1-\eta^i_m)I + \eta^i_m U_i$ for $i = 1, 2, ..., M_1$; $V^j_m = (1-\xi^j_m)I + \xi^j_m V_j$ for $j = 1, 2, ..., M_2$; $\eta \in (0,\ L^{-1})$ and L is the spectral radius of the operator A^*A.

Step 2. Compute

$$w_m = T^{(G_1,\phi_1)}_{r_m}(v_m + \eta A^*(T^{(G_2,\phi_2)}_{r_m} - I)Av_m).$$

Step 3. Compute

$$x_{m+1} = \sigma_m \gamma f(x_m) + b_m x_m + ((1-b_m) - \sigma_m \rho W)w_m, \qquad m \in \mathbb{N}. \tag{17}$$

Then the sequence $\{x_m\}$ strongly converges to $p^* \in \mathcal{F}$, which is the unique solution of the following variational inequality

$$\langle (\rho W - \gamma f)p^*, x - p^* \rangle \geq 0, \quad x \in \mathcal{F}, \tag{18}$$

or, equivalently, we have $P_{\mathcal{F}}(I - \rho W + \gamma f)p^* = p^*$.

We prove the convergence of Algorithm for $M_1 = M_2 = 2$ only. That is, we consider the case

$$v_m = U^m_2 U^m_1 J_{\lambda,E_1}(x_m + \eta A^*(V^m_2 V^m_1 J_{\lambda,E_2} - I)Ax_m).$$

Theorem 1 *Let $p \in \mathcal{F}$. Then the sequences $\{x_m\}$, $\{w_m\}$, $\{f(x_m)\}$ and $\{W(w_m)\}$ defined in Algorithm are bounded.*

Proof. From (B1), we may assume without loss of generality that

$$\sigma_m \rho \leq (1-b_m)\|W\|^{-1} \qquad \text{for all } m \geq 0.$$

Since W is a strongly positive bounded linear operator on $\mathcal{H}_1$,

$$\|W\| = \sup\{|\langle Wu, u\rangle| : u \in \mathcal{H}_1, \|u\| = 1\}.$$

Further,

$$\begin{aligned}\langle ((1-b_m)I - \sigma_m \rho W)u, u\rangle &= 1 - b_m - \sigma_m \rho \langle Wu, u\rangle \\ &\geq 1 - b_m - \sigma_m \rho \|W\| \geq 0,\end{aligned}$$

and hence,

$$\begin{aligned}&\|(((1-b_m)I - \sigma_m \rho W)u, u)\| \\ &\quad = \sup\{\langle ((1-b_m)I - \sigma_m \rho W)u, u\rangle : u \in \mathcal{H}_1, \|u\| = 1\} \\ &\quad = \sup\{1 - b_m - \sigma_m \rho \langle Wu, u\rangle : u \in \mathcal{H}_1, \|u\| = 1\} \\ &\quad \leq 1 - b_m - \sigma_m \rho \zeta.\end{aligned}$$

Let $p \in \mathcal{F}$ and $U_m = U_2^m U_1^m$, $V_m = V_2^m V_1^m$, $z_m = J_{\lambda,E_2} A x_m$, $y_m = x_m + \eta A^*(V_m J_{\lambda,E_2} - I)Ax_m$, $u_m = J_{\lambda,E_1} y_m$, $v_m = U_m u_m$ for all $m \in \mathbb{N}$. Since J_{λ,E_1} and J_{λ,E_2} are firmly non-expansive, they are also non-expansive, and we have

$$\|z_m - Ap\| = \|J_{\lambda,E_2} A x_m - Ap\| \leq \|Ax_m - Ap\|;$$
$$\|u_m - p\| = \|J_{\lambda,E_1} y_m - p\| \leq \|y_m - p\|.$$

Further, since $\xi_m^j, \eta_m^i \in (0,1)$, we conclude that U_i^m and V_j^m are averaged as a composition of averaged mappings.

Using the non-expansivity of the averaged mappings, we get

$$\begin{aligned}\|v_m - p\|^2 &= \|U_m J_{\lambda,E_1}(x_m + \eta A^*(V_m J_{\lambda,E_2} - I)Ax_m - p)\|^2 \\ &= \|U_m J_{\lambda,E_1}(x_m + \eta A^*(V_m J_{\lambda,E_2} - I)Ax_m - U_m J_{\lambda,E_1} p)\|^2 \\ &= \|x_m + \eta A^*(V_m J_{\lambda,E_2} - I)Ax_m - p)\|^2 \\ &= \|x_m - p\|^2 + \eta^2 \|A^*(V_m J_{\lambda,E_2} - I)Ax_m)\|^2 \\ &\quad + 2\eta\langle x_m - p, A^*(V_m J_{\lambda,E_2} - I)Ax_m\rangle.\end{aligned}$$

Due to firmly non-expansivity of J_{λ,E_2}, we obtain

$$\begin{aligned}&\langle x_m - p\ ,\ A^*(V_m J_{\lambda,E_2} - I)Ax_m\rangle = \langle Ax_m - Ap\ ,\ (V_m J_{\lambda,E_2} - I)Ax_m\rangle \\ &= \langle (V_m J_{\lambda,E_2} - I)Ax_m - Ap \\ &\quad + Ax_m - (V_m J_{\lambda,E_2} - I)Ax_m\ ,\ (V_m J_{\lambda,E_2} - I)Ax_m\rangle \\ &= \langle (V_m J_{\lambda,E_2} Ax_m - Ap\ ,\ (V_m J_{\lambda,E_2} - I)Ax_m)\rangle - \|(V_m J_{\lambda,E_2} - I)Ax_m)\|^2 \\ &= \frac{1}{2}\|V_m J_{\lambda,E_2} Ax_m - Ap\|^2 + \frac{1}{2}\|(V_m J_{\lambda,E_2} - I)Ax_m)\|^2 \\ &\quad - \frac{1}{2}\|Ax_m - Ap\|^2 - \|(V_m J_{\lambda,E_2} - I)Ax_m)\|^2 \\ &\leq \frac{1}{2}\|J_{\lambda,E_2} Ax_m - J_{\lambda,E_2} Ap\|^2 - \frac{1}{2}\|Ax_m - Ap\|^2 - \frac{1}{2}\|(V_m J_{\lambda,E_2} - I)Ax_m)\|^2 \\ &\leq \frac{1}{2}(\|Ax_m - Ap\|^2 - \|J_{\lambda,E_2} Ax_m - Ax_m\|^2) \\ &\quad - \frac{1}{2}\|Ax_m - Ap\|^2 - \frac{1}{2}\|(V_m J_{\lambda,E_2} - I)Ax_m)\|^2 \\ &= -\frac{1}{2}\|J_{\lambda,E_2} Ax_m - Ax_m\|^2 - \frac{1}{2}\|(V_m J_{\lambda,E_2} - I)Ax_m)\|^2 \\ &= -\frac{1}{2}\|z_m - Ax_m\|^2 - \frac{1}{2}\|(V_m J_{\lambda,E_2} - I)Ax_m)\|^2.\end{aligned}$$

Thus,

$$\begin{aligned}\|v_m - p\|^2 &\leq \|x_m - p\|^2 + \eta^2 L\|(V_m J_{\lambda,E_2} - I)Ax_m)\|^2 \\ &\quad - \eta\|(V_m J_{\lambda,E_2} - I)Ax_m)\|^2 - \|z_m - Ax_m\|^2 \\ &= \|x_m - p\|^2 + \eta(\eta L - 1)\|(V_m J_{\lambda,E_2} - I)Ax_m)\|^2 \\ &\quad - \|z_m - Ax_m\|^2. \end{aligned} \tag{19}$$

From the definition of η, we get

$$\|v_m - p\|^2 \leq \|x_m - p\|^2,$$

and hence,

$$\|v_m - p\| \leq \|x_m - p\|.$$

Since $p \in \mathcal{F}$, we have $T_{r_m}^{(G_1,\phi_1)} p = p$ and $T_{r_m}^{(G_2,\phi_2)} Ap = Ap$. Then

$$\begin{aligned}
\|w_m - p\|^2 &= \|T_{r_m}^{(G_1,\phi_1)}(v_m + \eta A^*(T_{r_m}^{(G_2,\phi_2)} - I)Av_m) - p\|^2 \\
&= \|T_{r_m}^{(G_1,\phi_1)}(v_m + \eta A^*(T_{r_m}^{(G_2,\phi_2)} - I)Av_m) - T_{r_m}^{(G_1,\phi_1)} p\|^2 \\
&\leq \|v_m + \eta A^*(T_{r_m}^{(G_2,\phi_2)} - I)Av_m) - p\|^2 \\
&\leq \|v_m - p\|^2 + \eta^2 \|A^*(T_{r_m}^{(G_2,\phi_2)} - I)Av_m)\|^2 \\
&\quad + 2\eta\langle v_m - p, A^*(T_{r_m}^{(G_2,\phi_2)} - I)Av_m)\rangle.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|w_m - p\|^2 \leq \|v_m - p\|^2 &+ \eta^2 \langle (T_{r_m}^{(G_2,\phi_2)} - I)Av_m \ , \ A^*A(T_{r_m}^{(G_2,\phi_2)} - I)Av_m\rangle \\
&+ 2\eta\langle v_m - p, A^*(T_{r_m}^{(G_2,\phi_2)} - I)Av_m)\rangle.
\end{aligned}$$

Further,

$$\begin{aligned}
&\eta^2 \langle (T_{r_m}^{(G_2,\phi_2)} - I)Av_m \ , \ A^*A(T_{r_m}^{(G_2,\phi_2)} - I)Av_m\rangle \\
&\leq \eta^2 L\langle (T_{r_m}^{(G_2,\phi_2)} - I)Av_m \ , \ (T_{r_m}^{(G_2,\phi_2)} - I)Av_m\rangle \\
&= \eta^2 L\|(T_{r_m}^{(G_2,\phi_2)} - I)Av_m)\|^2.
\end{aligned}$$

Using (13), we can write

$$\begin{aligned}
&2\eta\langle v_m - p \ , \ A^*(T_{r_m}^{(G_2,\phi_2)} - I)Av_m\rangle \\
&= 2\eta\langle A(v_m - p) \ , \ (T_{r_m}^{(G_2,\phi_2)} - I)Av_m\rangle \\
&= 2\eta\langle A(v_m - p) + (T_{r_m}^{(G_2,\phi_2)} - I)Av_m \\
&\quad - (T_{r_m}^{(G_2,\phi_2)} - I)Av_m \ , \ (T_{r_m}^{(G_2,\phi_2)} - I)Av_m\rangle \\
&= 2\eta\big\{\langle (T_{r_m}^{(G_2,\phi_2)} Av_m - Ap \ , \ (T_{r_m}^{(G_2,\phi_2)} - I)Av_m\rangle \\
&\quad - \|(T_{r_m}^{(G_2,\phi_2)} - I)Av_m\|^2\big\} \\
&\leq 2\eta\left\{\frac{1}{2}\|(T_{r_m}^{(G_2,\phi_2)} - I)Av_m\|^2 - \|(T_{r_m}^{(G_2,\phi_2)} - I)Av_m\|^2\right\} \\
&= -\eta\|(T_{r_m}^{(G_2,\phi_2)} - I)Av_m\|^2.
\end{aligned}$$

Thus, we obtain

$$\|w_m - p\|^2 \leq \|v_m - p\|^2 + \eta(\eta L - 1)\|(T_{r_m}^{(G_2,\phi_2)} - I)Av_m)\|^2.$$

From the definition of η, we obtain

$$\|w_m - p\| \le \|v_m - p\| \le \|x_m - p\|.$$

Since $0 < \sigma_m \rho < \|W\|^{-1}$, by Lemma 3, we get

$$\|I - \sigma_m \rho W\| \le 1 - \sigma_m \rho \zeta.$$

It follows that

$$\begin{aligned}
\|x_{m+1} - p\| &= \|\sigma_m \gamma f(x_m) + b_m x_m + ((1-b_m)I - \sigma_m \rho W)w_m - p\| \\
&= \|\sigma_m(\gamma f(x_m) - \rho W p) + b_m(x_m - p) \\
&\quad + ((1-b_m)I - \sigma_m \rho W)(w_m - p)\| \\
&\le \|\sigma_m(\gamma f(x_m) - \rho W p)\| + b_m\|x_m - p\| \\
&\quad + \|((1-b_m)I - \sigma_m \rho W)(w_m - p)\| \\
&\le \sigma_m\|\gamma f(x_m) - \gamma f(p)\| + \sigma_m\|\gamma f(x_m) - \rho W p\| \\
&\quad + b_m\|x_m - p\| + (1 - b_m - \sigma_m \rho \zeta)\|w_m - p\| \\
&\le \sigma_m \gamma \tau \|x_m - p\| + b_m\|x_m - p\| \\
&\quad + (1 - b_m - \sigma_m \rho \zeta)\|x_m - p\| + \sigma_m\|\gamma f(p) - \rho W p\| \\
&= [1 - \sigma_m(\rho\zeta - \gamma\tau)]\|x_m - p\| + \sigma_m(\rho\zeta - \gamma\tau)\frac{\|\gamma f(p) - \rho W p\|}{\rho\zeta - \gamma\tau}.
\end{aligned}$$

Continuing in the same way, we see that

$$\|x_{m+1} - p\| \le \max\left\{\|x_0 - p\|, \frac{\|\gamma f(p) - \rho W p\|}{\rho\zeta - \gamma\tau}\right\}, \quad m \ge 0, x_0 \in C.$$

Therefore, the sequence $\{x_m\}$ is bounded, and so are the sequences $\{v_m\}$, $\{w_m\}$, $\{y_m\}$, $\{z_m\}$, $\{f(x_m)\}$ and $\{W(w_m)\}$. $\square$

Theorem 2 *Let $p \in \mathcal{F}$ and $\{x_m\}$ be the sequence generated by Algorithm. Then*

(a) $\lim_{m\to\infty} \|x_{m+1} - x_m\| = 0$;

(b) Algorithm converges weakly to $p \in \mathcal{F}$.

Proof. (a) Consider $t_m = (x_{m+1} - b_m x_m)/(1 - b_m)$. Since

$$\begin{aligned}
t_{m+1} - t_m &= \frac{x_{m+2} - b_{m+1}x_{m+1}}{1 - b_{m+1}} - \frac{x_{m+1} - b_m x_m}{1 - b_m} \\
&= \frac{\sigma_{m+1}\gamma f(x_{m+1}) + ((1 - b_{m+1})I - \sigma_{m+1}\rho W)w_{m+1}}{1 - b_{m+1}} \\
&\quad - \frac{\sigma_m \gamma f(x_m) + ((1-b_m)I - \sigma_m \rho W)w_m}{1 - b_m} \\
&= \frac{\sigma_{m+1}(\gamma f(x_{m+1}) - \rho W w_{m+1})}{1 - b_{m+1}} - \frac{\sigma_m(\gamma f(x_m) - \rho W w_m)}{1 - b_m} + w_{m+1} - w_m,
\end{aligned}$$

we have

$$\begin{aligned}\|t_{m+1}-t_m\| &\le \frac{\sigma_{m+1}}{1-b_{m+1}}\|\gamma f(x_{m+1})-\rho W w_{m+1}\| \\ &\quad + \frac{\sigma_m}{1-b_m}\|\gamma f(x_m)-\rho W w_m\| + \|w_{m+1}-w_m\| \\ &\le \frac{\sigma_{m+1}}{1-b}\|\gamma f(x_{m+1})-\rho W w_{m+1}\| \\ &\quad + \frac{\sigma_m}{1-b}\|\gamma f(x_m)-\rho W w_m\| + \|w_{m+1}-w_m\|.\end{aligned}$$

Using Lemma 8, we can write

$$\|w_{m+1}-w_m\| = \|T^{(G_1,\phi_1)}_{r_{m+1}}(v_{m+1}+\eta A^*(T^{(G_2,\phi_2)}_{r_{m+1}}-I)Av_{m+1})$$

$$-T^{(G_1,\phi_1)}_{r_m}(v_m+\eta A^*(T^{(G_2,\phi_2)}_{r_m}-I)Av_m)\|$$

$$\le \|T^{(G_1,\phi_1)}_{r_{m+1}}(v_{m+1}+\eta A^*(T^{(G_2,\phi_2)}_{r_{m+1}}-I)Av_{m+1})$$

$$-T^{(G_1,\phi_1)}_{r_{m+1}}(v_m+\eta A^*(T^{(G_2,\phi_2)}_{r_m}-I)Av_m)\|$$

$$+\|T^{(G_1,\phi_1)}_{r_{m+1}}(v_m+\eta A^*(T^{(G_2,\phi_2)}_{r_m}-I)Av_m)$$

$$-T^{(G_1,\phi_1)}_{r_m}(v_m+\eta A^*(T^{(G_2,\phi_2)}_{r_m}-I)Av_m)\|$$

$$\le \|(v_{m+1}+\eta A^*(T^{(G_2,\phi_2)}_{r_{m+1}}-I)Av_{m+1})-(v_m+\eta A^*(T^{(G_2,\phi_2)}_{r_m}-I)Av_m)\|$$

$$+\left|\frac{r_{m+1}-r_m}{r_{m+1}}\right|\|T^{(G_1,\phi_1)}_{r_{m+1}}(v_m+\eta A^*(T^{(G_2,\phi_2)}_{r_m}-I)Av_m)$$

$$-(v_m+\eta A^*(T^{(G_2,\phi_2)}_{r_m}-I)Av_m)\|$$

$$\le \|v_{m+1}-v_m-\eta A^*A(v_{m+1}-v_m)\| + \eta\|A\|\|T^{(G_2,\phi_2)}_{r_{m+1}}Av_{m+1}-T^{(G_2,\phi_2)}_{r_m}Av_m\|$$

$$+s_m \le \left\{\|v_{m+1}-v_m\|^2-2\eta\|Av_{m+1}-Av_m\|^2+\eta^2\|A\|^4\|v_{m+1}-v_m\|^2\right\}^{1/2}$$

$$+\eta\|A\|\left\{\|Av_{m+1}-Av_m\|+\left|\frac{r_{m+1}-r_m}{r_{m+1}}\right|\|T^{(G_2,\phi_2)}_{r_{m+1}}Av_{m+1}-Av_{m+1}\|\right\}+s_m$$

$$\le (1-2\eta\|A\|^2+\eta^2\|A\|^4)^{\frac{1}{2}}\|v_{m+1}-v_m\|+\eta\|A\|^2\|v_{m+1}-v_m\|+\eta\|A\|\omega_m$$

$$+s_m = (1-\eta\|A\|^2)\|v_{m+1}-v_m\|+\eta\|A\|^2\|v_{m+1}-v_m\|+\eta\|A\|\omega_m+s_m$$

$$= \|v_{m+1}-v_m\|+\eta\|A\|\omega_m+s_m,$$

where

$$\omega_m = \left|\frac{r_{m+1} - r_m}{r_{m+1}}\right| \|T_{r_{m+1}}^{(G_2,\phi_2)} Av_{m+1} - Av_{m+1}\|$$

and

$$\begin{aligned} s_m = \left|\frac{r_{m+1} - r_m}{r_{m+1}}\right| &\|T_{r_{m+1}}^{(G_1,\phi_1)}(v_m + \eta A^*(T_{r_m}^{(G_2,\phi_2)} - I)Av_m) \\ &- (v_m + \eta A^*(T_{r_m}^{(G_2,\phi_2)} - I)Av_m)\|. \end{aligned}$$

Further,

$$\begin{aligned} \|v_{m+1} - v_m\| &= \|U_{m+1}u_{m+1} - U_m u_m\| \\ &= \|U_{m+1}u_{m+1} - U_{m+1}u_m + U_{m+1}u_m - U_m u_m\| \\ &\leq \|U_{m+1}u_{m+1} - U_{m+1}u_m\| + \|U_{m+1}u_m - U_m u_m\| \\ &\leq \|u_{m+1} - u_m\| + \|U_{m+1}u_m - U_m u_m\| \\ &\leq \|y_{m+1} - y_m\| + \|U_{m+1}u_m - U_m u_m\|. \end{aligned}$$

Since,

$$\begin{aligned} &\|y_{m+1} - y_m\| = \|x_{m+1} + \eta A^*(V_{m+1}J_{\lambda,E_2} - I)Ax_{m+1} \\ &\quad - x_m - \eta A^*(V_m J_{\lambda,E_2} - I)Ax_m\| \\ &\leq \|x_{m+1} - x_m - \eta A^* A(x_{m+1} - x_m)\| + \eta\|A\|\|V_{m+1}z_{m+1} - V_m z_m\| \\ &= \left\{\|x_{m+1} - x_m\|^2 - 2\eta\|Ax_{m+1} - Ax_m\|^2 + \eta^2\|A\|^4\|x_{m+1} - x_m\|^2\right\}^{1/2} \\ &\quad + \eta\|A\|\|V_{m+1}z_{m+1} - V_{m+1}z_m + V_{m+1}z_m - V_m z_m\| \\ &\leq (1 - \eta\|A\|^2)\|x_{m+1} - x_m\| \\ &\quad + \eta\|A\|\Big(\|V_{m+1}z_{m+1} - V_{m+1}z_m\| + \|V_{m+1}z_m - V_m z_m\|\Big) \\ &\leq (1 - \eta\|A\|^2)\|x_{m+1} - x_m\| + \eta\|A\|\Big(\|z_{m+1} - z_m\| + \|V_{m+1}z_m - V_m z_m\|\Big) \\ &\leq (1 - \eta\|A\|^2)\|x_{m+1} - x_m\| \\ &\quad + \eta\|A\|\Big(\|J_{\lambda,E_2}Ax_{m+1} - J_{\lambda,E_2}Ax_m\| + \|V_{m+1}z_m - V_m z_m\|\Big) \\ &\leq (1 - \eta\|A\|^2)\|x_{m+1} - x_m\| + \eta\|A\|\Big(\|Ax_{m+1} - Ax_m\| + \|V_{m+1}z_m - V_m z_m\|\Big) \\ &\leq \|x_{m+1} - x_m\| + \eta\|A\|\|V_{m+1}z_m - V_m z_m\|, \end{aligned}$$

we get

$$\begin{aligned} \|v_{m+1} - v_m\| \leq \|x_{m+1} - x_m\| &+ \eta\|A\|\|V_{m+1}z_m - V_m z_m\| \\ &+ \|U_{m+1}u_m - U_m u_m\|. \end{aligned}$$

Therefore,

$$\begin{aligned}&\|w_{m+1}-w_m\| \leq \|x_{m+1}-x_m\| + \eta\|A\|\|V_{m+1}z_m - V_m z_m\|\\&+\eta\|A\|\omega_m + \|U_{m+1}u_m - U_m u_m\| + s_m\\&= \|x_{m+1}-x_m\| + \eta\|A\|(\|V_{m+1}z_m - V_m z_m\| + \omega_m) + \|U_{m+1}u_m - U_m u_m\| + s_m.\end{aligned}$$

Substituting obtained estimations, we get

$$\begin{aligned}&\|t_{m+1}-t_m\| - \|x_{m+1}-x_m\|\\&\leq \frac{\sigma_{m+1}}{1-b}\|\gamma f(x_{m+1}) - \rho W w_{m+1}\| + \frac{\sigma_m}{1-b}\|\gamma f(x_m) - \rho W w_m\|\\&\quad + \eta\|A\|(\|V_{m+1}z_m - V_m z_m\| + \omega_m) + \|U_{m+1}u_m - U_m u_m\| + s_m.\end{aligned}$$

Further,

$$\begin{aligned}&\|U_{m+1}u_m - U_m u_m\| = \|U_2^{m+1}U_1^{m+1}u_m - U_2^m U_1^m u_m\|\\&\leq \|U_2^{m+1}U_1^{m+1}u_m - U_2^{m+1}U_1^m u_m\| + \|U_2^{m+1}U_1^m u_m - U_2^m U_1^m u_m\|.\end{aligned}$$

It follows from the definition of U_i^m that

$$\begin{aligned}&\|U_1^{m+1}u_m - U_1^m u_m\|\\&= \|(1-\eta_{m+1}^1)u_m + \eta_{m+1}^1)U_1 u_m - (1-\eta_m^1)u_m + \eta_m^1)U_1 u_m\|\\&\leq |\eta_{m+1}^1 - \eta_m^1|(\|u_m\| + \|U_1 u_m\|).\end{aligned}$$

Since $\lim_{m\to\infty} |\eta_{m+1}^1 - \eta_m^1| = 0$ for $i = 1, 2$ and the sequences $\{u_m\}$ and $\{U_1 u_m\}$ are bounded, we get

$$\lim_{m\to\infty} \|U_1^{m+1}u_m - U_1^m u_m\| = 0.$$

Similarly,

$$\|U_2^{m+1}U_1^m u_m - U_2^m U_1^m u_m\| \leq |\eta_{m+1}^2 - \eta_m^2|(\|U_1^m u_m\| + \|U_2^m U_1^m u_m\|),$$

from which it follows that

$$\lim_{m\to\infty} \|U_2^{m+1}U_1^m u_m - U_2^m U_1^m u_m\| = 0.$$

Therefore,

$$\lim_{m\to\infty} \|U_{m+1}u_m - U_m u_m\| = 0. \tag{20}$$

Using the similar reasoning, one can show that

$$\lim_{m\to\infty} \|V_{m+1}u_m - V_m u_m\| = 0. \tag{21}$$

Hence, due to (20), (21) and (B1),(B2), we get

$$\lim_{m\to\infty} (\|t_{m+1}-t_m\| - \|x_{m+1}-x_m\|) \leq 0.$$

Thus, by Lemma 10, we conclude that $\lim_{m\to\infty} \|t_m - x_m\| = 0$, which implies that

$$\lim_{m\to\infty} \|x_{m+1} - x_m\| = 0. \tag{22}$$

(b) To show the weak convergence of Algorithm, consider

$$\begin{aligned}
&\|x_m - w_m\| \le \|x_m - x_{m+1}\| + \|x_{m+1} - w_m\| \\
&\le \|x_m - x_{m+1}\| + \|\sigma_m \gamma f(x_m) + b_m x_m + ((1-b_m)I - \sigma_m \rho W)w_m - w_m\| \\
&\le \|x_m - x_{m+1}\| + \sigma_m \|\gamma f(x_m) - \rho W w_m\| + b_m \|x_m - w_m\| \\
&\le \|x_m - x_{m+1}\| + \sigma_m(\|\gamma f(x_m)\| + \|\rho W w_m\|) + b_m \|x_m - w_m\|.
\end{aligned}$$

That is,

$$\|x_m - w_m\| \le \frac{1}{1-b_m}\|x_m - x_{m+1}\| + \frac{\sigma_m}{1-b_m}(\|\gamma f(x_m)\| + \|\rho W w_m\|),$$

which together with (B1) implies that

$$\lim_{m\to\infty} \|x_m - w_m\| = 0. \tag{23}$$

Now let us show that $\lim_{m\to\infty} \|x_m - v_m\| = 0$. Using the non-expansivity of averaged mappings, we can write

$$\begin{aligned}
&\|v_m - p\|^2 = \|U_m J_{\lambda,E_1}(x_m + \eta A^*(V_m J_{\lambda,E_2} - I)Ax_m) - p\|^2 \\
&= \|U_m J_{\lambda,E_1}(x_m + \eta A^*(V_m J_{\lambda,E_2} - I)Ax_m) - U_m J_{\lambda,E_1} p\|^2 \\
&= \|x_m + \eta A^*(V_m J_{\lambda,E_2} - I)Ax_m - p\|^2 \\
&\le \langle v_m - p, x_m + \eta A^*(V_m J_{\lambda,E_2} - I)Ax_m - p\rangle \\
&= \frac{1}{2}\Big\{ \|v_m - p\|^2 + \|x_m + \eta A^*(V_m J_{\lambda,E_2} - I)Ax_m - p\|^2 \\
&\quad - \|(v_m - p) - (x_m + \eta A^*(V_m J_{\lambda,E_2} - I)Ax_m - p)\|^2 \Big\} \\
&= \frac{1}{2}\Big\{ \|v_m - p\|^2 + \|x_m - p\|^2 + \eta(L\eta - 1)(V_m J_{\lambda,E_2} - I)Ax_m\|^2 \\
&\quad - \|v_m - x_m - \eta A^*(V_m J_{\lambda,E_2} - I)Ax_m\|^2 \Big\} \\
&= \frac{1}{2}\Big\{ \|v_m - p\|^2 + \|x_m - p\|^2 - [\|v_m - x_m\|^2 \\
&\quad + \eta^2 \|A^*(V_m J_{\lambda,E_2} - I)Ax_m\|^2 - 2\eta \langle v_m - x_m, A^*(V_m J_{\lambda,E_2} - I)Ax_m\rangle] \Big\} \\
&= \frac{1}{2}\Big\{ \|v_m - p\|^2 + \|x_m - p\|^2 - \|v_m - x_m\|^2 \\
&\quad + 2\eta \|A(v_m - x_m)\| \|(V_m J_{\lambda,E_2} - I)Ax_m\| \Big\}.
\end{aligned}$$

Hence,

$$\|v_m - p\|^2 \leq \|x_m - p\|^2 - \|v_m - x_m\|^2 \\ + 2\eta\|A(v_m - x_m)\|\|(V_m J_{\lambda,E_2} - I)Ax_m\|. \quad (24)$$

Now we consider $\|x_{m+1} - p\|^2$. Using (15) and (16), we can write

$$\begin{aligned}
&\|x_{m+1} - p\|^2 = \|\sigma_m \gamma f(x_m) + b_m x_m + ((1 - b_m)I - \sigma_m \rho W)w_m - p\|^2 \\
&= \|\sigma_m(\gamma f(x_m) - \rho W p) + b_m(x_m - w_m) + (I - \sigma_m \rho W)(w_m - p)\|^2 \\
&\leq \|b_m(x_m - w_m) + (I - \sigma_m \rho W)(w_m - p)\|^2 \\
&\quad + 2\sigma_m \langle \gamma f(x_m) - \rho W p, x_{m+1} - p\rangle \\
&\leq \left[(1 - \sigma_m \rho\zeta)\|w_m - p\| + b_m\|x_m - w_m\|^2\right] \\
&\quad + 2\sigma_m\|\gamma f(x_m) - \rho W p\|\|x_{m+1} - p\| \\
&\leq (1 - \sigma_m \rho\zeta)^2\|w_m - p\|^2 + b_m^2\|x_m - w_m\|^2 \\
&\quad + 2b_m(1 - \sigma_m \rho\zeta)\|w_m - p\|\|x_m - w_m\| \\
&\quad + 2\sigma_m\|\gamma f(x_m) - \rho W p\|\|x_{m+1} - p\|.
\end{aligned}$$

Using (**??**), we obtain

$$\begin{aligned}
&\|x_{m+1} - p\|^2 \leq (1 - \sigma_m \rho\zeta)^2\left[\|v_m - p\|^2 + \eta(\eta L - 1)\|(T_{r_m}^{(G_2,\phi_2)} - I)Av_m)\|^2\right] \\
&\quad + b_m^2\|x_m - w_m\|^2 + 2b_m(1 - \sigma_m \rho\zeta)\|w_m - p\|\|x_m - w_m\| \\
&\quad + 2\sigma_m\|\gamma f(x_m) - \rho W p\|\|x_{m+1} - p\| \\
&\leq (1 - \sigma_m \rho\zeta)^2\left[\|x_m - p\|^2 + \eta(\eta L - 1)\|(T_{r_m}^{(G_2,\phi_2)} - I)Av_m)\|^2\right] \\
&\quad + b_m^2\|x_m - w_m\|^2 + 2b_m(1 - \sigma_m \rho\zeta)\|w_m - p\|\|x_m - w_m\| \\
&\quad + 2\sigma_m\|\gamma f(x_m) - \rho W p\|\|x_{m+1} - p\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&(1 - \sigma_m \rho\zeta)^2\eta(1 - \eta L)\|(T_{r_m}^{(G_2,\phi_2)} - I)Av_m)\|^2 \\
&\leq b_m^2\|x_m - w_m\|^2 + 2b_m(1 - \sigma_m \rho\zeta)\|w_m - p\|\|x_m - w_m\| \\
&\quad + 2\sigma_m\|\gamma f(x_m) - \rho W p\|\|x_{m+1} - p\| \\
&\quad + (1 - \sigma_m \rho\zeta)^2\|x_m - p\|^2 - \|x_{m+1} - p\|,
\end{aligned}$$

which gives

$$\begin{aligned}
&(1 - \sigma_m \rho\zeta)^2\eta(1 - \eta L)\|(T_{r_m}^{(G_2,\phi_2)} - I)Av_m)\|^2 \\
&\leq (\sigma_m \rho\zeta)^2\|x_m - p\|^2 + 2b_m(1 - \sigma_m \rho\zeta)\|w_m - p\|\|x_m - w_m\| \\
&\quad + 2\sigma_m\|\gamma f(x_m) - \rho W p\|\|x_{m+1} - p\| + b_m^2\|x_m - w_m\|^2 \\
&\quad - 2\sigma_m \rho\zeta\|x_m - p\|^2 + \|x_m - x_{m+1}\|(\|x_m - p\| + \|x_{m+1} - p\|).
\end{aligned}$$

Further, using (19), we can write

$$\begin{aligned}\|x_{m+1}-p\|^2 \leq (1-\sigma_m\rho\zeta)^2\Big[&\|x_m-p\|^2+\eta(\eta L-1)\|(V_mJ_{\lambda,E_2}-I)Ax_m)\|^2\\ &-\|z_m-Ax_m\|^2+\eta(\eta L-1)\|(T_{r_m}^{(G_2,\phi_2)}-I)Av_m)\|^2\Big]\\ &+b_m^2\|x_m-w_m\|^2+2b_m(1-\sigma_m\rho\zeta)\|w_m-p\|\|x_m-w_m\|\\ &+2\sigma_m\|\gamma f(x_m)-\rho Wp\|\|x_{m+1}-p\|.\end{aligned}$$

Since due to (B3), (22) and (23), we get

$$\lim_{m\to\infty}\|(T_{r_m}^{(G_2,\phi_2)}-I)Av_m)\|=0, \tag{25}$$

we obtain

$$\begin{aligned}\|x_{m+1}-p\|^2 \leq (1-\sigma_m\rho\zeta)^2\Big[&\|x_m-p\|^2+\eta(\eta L-1)\|(V_mz_m-Ax_m)\|^2\\ &-\|z_m-Ax_m\|^2\Big]+b_m^2\|x_m-w_m\|^2\\ &+2b_m(1-\sigma_m\rho\zeta)\|w_m-p\|\|x_m-w_m\|\\ &+2\sigma_m\|\gamma f(x_m)-\rho Wp\|\|x_{m+1}-p\|.\end{aligned}$$

Hence,

$$\begin{aligned}&(1-\sigma_m\rho\zeta)^2\big[\eta(\eta L-1)\|(V_mz_m-Ax_m)\|^2-\|z_m-Ax_m\|^2\big]\\ &\leq b_m^2\|x_m-w_m\|^2+2b_m(1-\sigma_m\rho\zeta)\|w_m-p\|\|x_m-w_m\|\\ &\quad+2\sigma_m\|\gamma f(x_m)-\rho Wp\|\|x_{m+1}-p\|+(1-\sigma_m\rho\zeta)^2\|x_m-p\|^2\\ &\quad-\|x_{m+1}-p\|^2.\end{aligned}$$

Thus, from (23) and (B2), we get

$$\lim_{m\to\infty}\big[\|V_mz_m-Ax_m\|^2+\|z_m-Ax_m\|^2\big]=0,$$

which means that

$$\lim_{m\to\infty}\|V_mz_m-Ax_m\|=0. \tag{26}$$

Due to (24), we have

$$\begin{aligned}\|x_{m+1}-p\|^2 \leq (1-\sigma_m\rho\zeta)^2\Big[&\|x_m-p\|^2-\|v_m-x_m\|^2\\ &+2\eta\|A(v_m-x_m)\|\|V_mz_m-Ax_m\|\Big]+b_m^2\|x_m-w_m\|^2\\ &+2b_m(1-\sigma_m\rho\zeta)\|w_m-p\|\|x_m-w_m\|\\ &+2\sigma_m\|\gamma f(x_m)-\rho Wp\|\|x_{m+1}-p\|,\end{aligned}$$

and therefore,

$$\begin{aligned}
&(1-\sigma_m\rho\zeta)^2\|v_m-x_m\|^2\\
&\le 2\eta(1-\sigma_m\rho\zeta)^2\|A(v_m-x_m)\|\|V_mz_m-Ax_m\|\\
&\quad + b_m^2\|x_m-w_m\|^2+2b_m(1-\sigma_m\rho\zeta)\|w_m-p\|\|x_m-w_m\|\\
&\quad + 2\sigma_m\|\gamma f(x_m)-\rho Wp\|\|x_{m+1}-p\|+(\sigma_m\gamma\zeta)^2\|x_m-p\|^2\\
&\quad - 2\sigma_m\rho\zeta\|x_m-p\|^2+\|x_{m+1}-x_m\|(\|x_m-p\|+\|x_{m+1}-p\|).
\end{aligned}$$

Thus, from (B3), (23) and (26), we get

$$\lim_{m\to\infty}\|v_m-x_m\|=0. \tag{27}$$

Now, we can write

$$\begin{aligned}
\|x_m-U_mu_m\| &\le \|x_m-x_{m+1}\|+\|x_{m+1}-U_mu_m\|\\
&\le \|x_m-x_{m+1}\|+\|\sigma_m\gamma f(x_m)+b_mx_m\\
&\quad + ((I-b_m)-\sigma_m\rho W)w_m-U_mu_m\|\\
&\le \|x_m-x_{m+1}\|+\sigma_m\|\gamma f(x_m)-\rho Ww_m\|\\
&\quad + b_m\|x_m-u_mu_m\|\\
&\le \|x_m-x_{m+1}\|+\sigma_m(\|\gamma f(x_m)\|+\|\rho Ww_m\|)\\
&\quad + b_m\|x_m-u_mu_m\|,
\end{aligned}$$

that is

$$\begin{aligned}
\|x_m-U_mu_m\| \le \frac{1}{1-b_m}&\|x_m-x_{m+1}\|\\
&+\frac{\sigma_m}{1-b_m}(\|\gamma f(x_m)\|+\|\rho Ww_m\|),
\end{aligned}$$

which together with (B1) implies that

$$\lim_{m\to\infty}\|x_m-U_mu_m\|=0. \tag{28}$$

From (27), we get

$$\|x_{m+1}-v_m\|\le\|x_{m+1}-x_m\|+\|x_m-v_m\|\to 0 \quad \text{as} \quad m\to\infty. \tag{29}$$

From 22, (23) and (29), we get

$$\|w_m-v_m\|\le\|w_m-x_m\|+\|x_m-x_{m+1}\|+\|x_{m+1}-v_m\|\to 0$$

as $m\to\infty$. Further,

$$\begin{aligned}
\|x_m-U_mx_m\| &\le \|x_m-U_mu_m\|+\|U_mu_m-U_mw_m\|+\|U_mw_m-U_mx_m\|\\
&\le \|x_m-U_mu_m\|+\|v_m-U_mw_m\|+\|w_m-x_m\|.
\end{aligned}$$

From here it follows that

$$\lim_{n\to\infty} \|x_m - U_m w_m\| = 0. \tag{30}$$

Finally, using equation (28) and (30), we get

$$\|x_m - U_m x_m\| \leq \|x_m - U_m u_m\| + \|v_m - U_m w_m\| + \|w_m - x_m\| \to 0$$

as $m \to \infty$. □

Theorem 3 *The sequence $\{x_m\}$ generated by Algorithm strongly converges to p^*, which is the unique solution to the variational inequality* (18).

Proof. First, we show that the solution of the variational inequality problem (18) is unique. Suppose towards a contradiction that $\hat{p}$ and $\tilde{p}$ are two different solutions to variational inequality problem (18). Then

$$\langle(\rho W - \gamma f)\tilde{p}, \tilde{p} - \hat{p}\rangle \leq 0 \quad \text{and} \quad \langle(\rho W - \gamma f)\hat{p}, \hat{p} - \tilde{p}\rangle \leq 0.$$

Adding the above two inequalities, we get

$$\langle(\rho W - \gamma f)\tilde{p} - (\rho W - \gamma f)\hat{p}, \tilde{p} - \hat{p}\rangle \leq 0.$$

According to (B1), $\rho\zeta > \gamma\tau$. Thus, by Lemma 5, we get

$$\langle(\rho W - \gamma f)\tilde{p} - (\rho W - \gamma f)\hat{p}, \tilde{p} - \hat{p}\rangle \geq (\rho\zeta - \gamma\tau)\|\tilde{p} - \hat{p}\|^2 > 0,$$

which leads to the contradiction. Hence, the variational inequality problem (18) has a unique solution $p^* \in \mathcal{F}$.

Now, we show that

$$\lim_{m\to\infty} \sup\langle(\rho W - \gamma f)p^*, p^* - x_m)\rangle \leq 0.$$

Due to the boundedness of $\{x_m\}$, there exists a subsequence $\{x_{m_j}\}$ of $\{x_m\}$ such that $x_{m_j} \rightharpoonup \bar{p}$ as $j \to \infty$ and

$$\begin{aligned}&\lim_{m\to\infty} \sup\langle(\rho W - \gamma f)p^*, p^* - x_m)\rangle \\ &= \lim_{m\to\infty} \sup\langle(\rho W - \gamma f)p^*, p^* - x_{m_j})\rangle.\end{aligned}$$

Since $\|x_m - y_m\| \to 0$ as $m \to \infty$, we have $y_{m_j} \rightharpoonup \bar{p}$. Note that $\{\eta_m^i\}$ is bounded for $i = 1, 2$. We can assume that $\eta_{m_j}^i \to \eta_\infty^i$ as $j \to \infty$ where $0 < \eta_\infty^i < 1$ for $i = 1, 2$.

Define

$$U_i^\infty = (1 - \eta_\infty^i)I + \eta_\infty^i U_i, \qquad i = 1, 2.$$

Then we have

$$\mathrm{Fix}(U_i^\infty) = \mathrm{Fix}(U_i), \qquad i = 1, 2.$$

Furthermore, since

$$\mathrm{Fix}(U_1^\infty) \cap \mathrm{Fix}(U_2^\infty) = \mathrm{Fix}(U_1)\mathrm{Fix}(U_2) = \emptyset.$$

and U_i^∞ is η_∞^i-averaged for $i = 1, 2$, by Lemma 1, we get

$$\mathrm{Fix}(U_2^\infty U_1^\infty) = \mathrm{Fix}(U_1^\infty)\mathrm{Fix}(U_2^\infty) = \mathrm{Fix}(U_1) \cap \mathrm{Fix}(U_2).$$

Note that

$$\|U_i^{m_j} u - U_i^\infty u\| \le |\eta_{m_j}^i - \eta_\infty^i|(\|u\| + \|U_i u\|).$$

Hence,

$$\lim_{j\to\infty} \sup_{u\in\mathcal{B}} \|U_i^{m_j} u - U_i^\infty u\| = 0,$$

where $\mathcal{B}$ is an arbitrary bounded subset of $\mathcal{H}_1$. Also, we have

$$\begin{aligned}
\|x_{m_j} - U_2^\infty U_1^\infty x_{m_j}\| &\le \|x_{m_j} - U_2^{m_j} U_1^{m_j} x_{m_j}\| + \|U_2^{m_j} U_1^{m_j} x_{m_j} - U_2^\infty U_1^{m_j} x_{m_j}\| \\
&\quad + \|U_2^\infty U_1^{m_j} x_{m_j} - U_2^\infty U_1^\infty x_{m_j}\| \\
&\le \|x_{m_j} - U_2^{m_j} U_1^{m_j} x_{m_j}\| + \|U_2^{m_j} U_1^{m_j} x_{m_j} - U_2^\infty U_1^{m_j} x_{m_j}\| \\
&\quad + \|U_1^{m_j} x_{m_j} - U_1^\infty x_{m_j}\| \\
&\le \|x_{m_j} - U_2^{m_j} U_1^{m_j} x_{m_j}\| + \sup_{x\in\mathcal{B}'} \|U_2^{m_j} x - U_2^\infty x\| \\
&\quad + \sup_{x\in\mathcal{B}''} \|U_1^{m_j} x - U_1^\infty x\|,
\end{aligned}$$

where $\mathcal{B}'$ is a bounded subset including $\{U_1^{m_j} x_{m_j}\}$, and $\mathcal{B}''$ is a bounded subset including $\{x_{m_j}\}$. It follows that

$$\lim_{j\to\infty} \|x_{m_j} - U_2^\infty U_1^\infty x_{m_j}\| = 0.$$

Thus, by Lemma 2, we have

$$\bar{p} \in \mathrm{Fix}(U_2^\infty U_1^\infty) = \mathrm{Fix}(U_1) \cap \mathrm{Fix}(U_2).$$

Now, $y_{m_j} \rightharpoonup \bar{p}$, $u_{m_j} \rightharpoonup \bar{p}$, $Ax_m \rightharpoonup A\bar{p}$, and $z_{m_j} \rightharpoonup A\bar{p}$. Using the above arguments, we can show that

$$\lim_{j\to\infty} \|z_{m_j} - V_2^\infty V_1^\infty z_{m_j}\| = 0.$$

Since $V_2^\infty V_1^\infty$ is non-expansive, by Lemma 2 we get

$$A\bar{p} \in \mathrm{Fix}(V_2^\infty V_1^\infty) = \mathrm{Fix}(V_1) \cap \mathrm{Fix}(V_2).$$

Next, we show that $\bar{p} \in \mathrm{SOL}(E_1)$ and $A\bar{p} \in \mathrm{SOL}(E_2)$. Since $y_{m_j} \rightharpoonup \bar{p}$,

$$\lim_{m\to\infty} \|u_m - y_m\| = \lim_{m\to\infty} \|J_{\lambda,E_1} y_m - y_m\| = 0,$$

and J_{λ,E_1} is non-expansive, by Lemma 2, we get $\bar{p} = J_{\lambda,E_1}\bar{p}$, i.e., $\bar{p} \in \mathrm{SOL}(E_1)$. Also, since $z_{m_j} \rightharpoonup A\bar{p}$,

$$\lim_{m\to\infty} \|z_m - Ax_m\| = \lim_{m\to\infty} \|J_{\lambda,E_2} Ax_m - Ax_m\| = 0,$$

and by Lemma 2, we get $A\bar{p} = J_{\lambda,E_2}A\bar{p}$, i.e., $A\bar{p} \in \mathrm{SOL}(E_2)$. Therefore, $\bar{p} \in \Omega$.

Now, we show that $\bar{p} \in \mathcal{S}$. First, we will show that $\bar{p} \in \mathrm{GEP}(G_1, \phi_1)$. Since $w_m = T_{r_m}^{(G_1,\phi_1)} v_m$, we have

$$G_1(w_m, y) + \phi_1(w_m, y) + \frac{1}{r_m}\langle y - w_m, w_m - v_m\rangle \geq 0, \qquad y \in C.$$

It follows from the monotonicity of G_1 that

$$\phi_1(w_m, y) + \frac{1}{r_m}\langle y - w_m, w_m - v_m\rangle \geq G_1(w_m, y),$$

and hence, replacing m by m_i, we get

$$\phi_1(w_{m_i}, y) + \left\langle y - w_{m_i}, \frac{w_{m_i} - v_{m_i}}{r_{m_i}} \right\rangle \geq G_1(w_{m_i}, y).$$

Since $\|w_m - v_m\| \to 0$, we have $w_{m_i} \rightharpoonup \bar{p}$ and $\dfrac{w_{m_i} - v_{m_i}}{r_{m_i}} \to 0$. It follows from (iii) in Assumptions 1, that $0 \geq G_1(w_m, y)$ for any $\bar{p} \in C$. For any t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)\bar{p}$. Since $y, \bar{p} \in C$, we have $y_t \in C$, and hence, $G_1(y_t, \bar{p}) \leq 0$. Thus, due to (i) and (iv) in Assumptions 1, we obtain

$$\begin{aligned} 0 &= G_1(y_t, y_t) + \phi_1(y_t, y_t) \\ &\leq t[G_1(y_t, y) + \phi_1(y_t, y)] + (1-t)[G_1(y_t, \bar{p}) + \phi_1(y_t, \bar{p})] \\ &\leq t[G_1(y_t, y) + \phi_1(y_t, y)] + (1-t)[G_1(\bar{p}, y_t) + \phi_1(\bar{p}, y_t)] \\ &\leq t[G_1(y_t, y) + \phi_1(y_t, y)]. \end{aligned}$$

Therefore, $0 \leq G_1(y_t, y) + \phi_1(y_t, y)$. From Assumptions 1 (iii), we have $0 \leq G_1(\bar{p}, y) + \phi_1(\bar{p}, y)$. This implies that $\bar{p} \in \mathrm{GEP}(G_1, \phi_1)$.

Next, we show that $A\bar{p} \in \mathrm{GEP}(G_2, \phi_2)$. Since $\|w_m - v_m\| \to 0$, $\|v_m - x_m\| \to 0$, we get $w_m \rightharpoonup \bar{p}$ as $m \to \infty$ and $\{x_m\}$ is bounded, there exists a subsequence $\{x_{m_i}\}$ such that $x_{m_i} \rightharpoonup \bar{p}$, and since A is bounded linear operator, $Ax_{m_i} \rightharpoonup A\bar{p}$.

Now, set $q_{m_i} = Ax_{m_i} - T_{r_{m_i}}^{(G_2,\phi_2)} Ax_{m_i}$. It follows from (25), that

$$\lim_{i\to\infty} q_{m_i} = 0 \qquad \text{and} \qquad Ax_{m_i} - q_{m_i} = T_{r_{m_i}}^{(G_2,\phi_2)} Ax_{m_i}.$$

Therefore, from Lemma 7, we have

$$\begin{aligned} &G_2(Ax_{m_i} - q_{m_i}, \tilde{\bar{p}}) + \phi_2(Ax_{m_i} - q_{m_i}, \tilde{\bar{p}}) \\ &\quad + \frac{1}{r_{m_i}} \big\langle \tilde{\bar{p}} - (Ax_{m_i} - q_{m_i}), (Ax_{m_i} - q_{m_i}) - Ax_{m_i} \big\rangle \geq 0 \end{aligned} \tag{31}$$

for all $\tilde{\bar{p}} \in Q$. Since G_2 and ϕ_2 are upper semi continuous, taking limit superior in (31) as $i \to \infty$ and using condition (B3), we obtain

$$G_2(A\bar{p}, \tilde{\bar{p}}) + \phi_2(A\bar{p}, \tilde{\bar{p}}) \geq 0, \qquad \tilde{\bar{p}} \in Q,$$

which implies that $A\bar{p} \in \mathrm{GEP}(G_2, \phi_2)$ and hence, $\bar{p} \in \mathcal{S}$. Therefore, $\bar{p} \in \mathcal{F}$. Thus,

$$\lim_{j\to\infty} \langle (\rho W - \gamma f)p^*, p^* - x_{m_j} \rangle \leq \langle (\rho W - \gamma f)p^*, p^* - \bar{p} \rangle \leq 0. \tag{32}$$

Finally, we show that $x_m \to p^*$ as $m \to \infty$. From (15) and (17), we have

$$\begin{aligned} \|x_{m+1} - p^*\|^2 &= \|\sigma_m \gamma f(x_m) + b_m x_m + ((1-b_m)I - \sigma_m \rho W)w_m - p^*\|^2 \\ &= \|\sigma_m(\gamma f(x_m) - \rho W p^*) + b_m(x_m - p^*) \\ &\quad + ((1-b_m)I - \sigma_m \rho W)(w_m - p^*)\|^2 \\ &\leq \|b_m(x_m - p^*) + ((1-b_m)I - \sigma_m \rho W)(w_m - p^*)\|^2 \\ &\quad + 2\sigma_m \langle \gamma f(x_m) - \rho W p^*, x_{m+1} - p^* \rangle \\ &\leq \big[(1 - b_m - \sigma_m \rho\zeta)\|w_m - p^*\| + b_m\|x_m - p^*\|\big]^2 \\ &\quad + 2\sigma_m \langle \gamma f(x_m) - \gamma f(p^*), x_{m+1} - p^* \rangle \\ &\quad + 2\sigma_m \langle \gamma f(p^*) - \rho W p^*, x_{m+1} - p^* \rangle \\ &\leq \big[(1 - b_m - \sigma_m \rho\zeta)\|x_m - p^*\| + b_m\|x_m - p^*\|\big]^2 \\ &\quad + 2\sigma_m \gamma\tau \|x_m - p^*\|\|x_{m+1} - p^*\| \\ &\quad + 2\sigma_m \langle \gamma f(p^*) - \rho W p^*, x_{m+1} - p^* \rangle \\ &\leq 2\sigma_m \langle (\rho W - \gamma f)p^*, p^* - x_{m+1} \rangle. \end{aligned}$$

Since $\rho\zeta > \gamma\tau$ and $0 < \sigma_m \leq \dfrac{1}{\rho\|W\|} \leq \dfrac{1}{\rho\zeta}$, we get

$$1 - \sigma_m \gamma\tau > 1 - \sigma_m \rho\zeta \geq 0.$$

Hence, we can write

$$\begin{aligned}
\|x_{m+1}-p^*\| &\leq \frac{(1-\sigma_m\rho\zeta)^2+\sigma_m\gamma\tau}{1-\sigma_m\gamma\tau}\|x_m-p^*\|^2 \\
&\quad + \frac{2\sigma_m}{1-\sigma_m\gamma\tau}\langle(\rho W-\gamma f)p^*, p^*-x_{m+1}\rangle \\
&\leq \left[1-\frac{2\sigma_m(\rho\zeta-\gamma\tau)}{1-\sigma_m\gamma\tau}\right]\|x_m-p^*\|^2+\frac{\sigma_m^2\rho^2\zeta^2}{1-\sigma_m\gamma\tau}\|x_m-p^*\| \\
&\quad + \frac{2\sigma_m}{1-\sigma_m\gamma\tau}\langle(\rho W-\gamma f)p^*, p^*-x_{m+1}\rangle \\
&\leq \left[1-\frac{2\sigma_m(\rho\zeta-\gamma\tau)}{1-\sigma_m\gamma\tau}\right]\|x_m-p^*\|^2 \\
&\quad + \frac{2\sigma_m(\rho\zeta-\gamma\tau)}{1-\sigma_m\gamma\tau}\left[\frac{\langle(\rho W-\gamma f)p^*, p^*-x_{m+1}\rangle}{\rho\zeta-\gamma\tau}+\sigma_m L\right],
\end{aligned}$$

where L is a constant satisfying

$$L \geq \sup_{m\geq 0}\left\{\frac{\rho^2\zeta^2}{2}\|x_m-p^*\|^2\right\}.$$

Now, using the condition (B3) and (32), we obtain

$$\sum_{m=0}^{\infty}\frac{2\sigma_m(\rho\zeta-\gamma\tau)}{1-\sigma_m\gamma\tau} > \sum_{m=0}^{\infty}2(\rho\zeta-\gamma\tau)\sigma_m=\infty,$$

and

$$\limsup_{m\to\infty}\left(\frac{\langle(\rho W-\gamma f)p^*, p^*-x_{m+1}\rangle}{\rho\zeta-\gamma\tau}+\sigma_m L\right)\leq 0.$$

Therefore, according to Lemma 9, $\|x_m-p^*\|\to 0$ as $m\to\infty$.

To conclude the proof, note that the variational inequality (18) can be rewritten as

$$\langle(I-\rho W+\gamma f)p^*-p^*, x-p^*\rangle\leq 0, \qquad x\in\mathcal{F},$$

which, due to (14), is equivalent to the fixed point equation

$$P_{\mathcal{F}}(I-\rho W+\gamma f)p^*=p^*.$$

□

3 Numerical Example

Set $\mathcal{H}_1=\mathcal{H}_2=\mathbb{R}$. Let $C=[0,+\infty)$ and $Q=(-\infty,0]$. Suppose $A:\mathbb{R}\to\mathbb{R}$, $W:\mathbb{R}\to\mathbb{R}$, $U:C\to C$, $V:Q\to Q$ and $f:\mathbb{R}\to\mathbb{R}$ are defined by

Table 1: Results for different initial values

No. of iterations	$x_0 = -1.5$	$x_0 = 1.0$	$x_0 = 2.0$
1	-1.500000	1.000000	2.000000
2	-0.254372	0.170616	0.336936
3	-0.049675	0.033328	0.065775
4	-0.010527	0.007063	0.013939
5	-0.002359	0.001583	0.003124
6	-0.000551	0.000370	0.000730
7	-0.000133	0.000089	0.000176
8	-0.000033	0.000022	0.000043
9	-0.000008	0.000006	0.000011
10	-0.000002	0.000001	0.000003
11	-0.000001	0.000000	0.000001
12	-0.000000	0.000000	0.000000

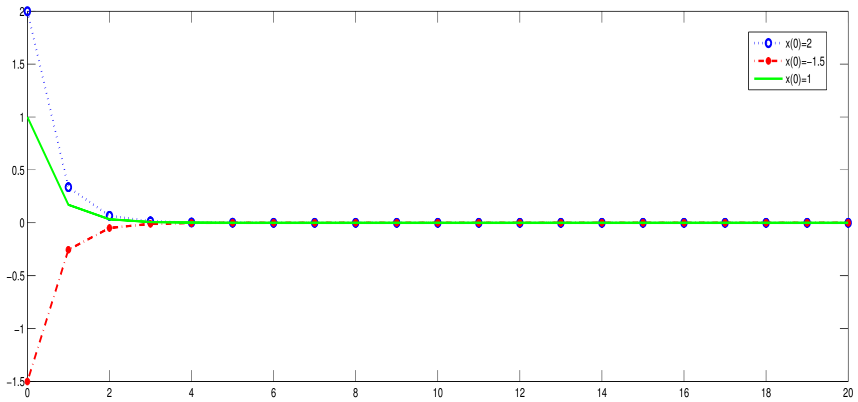

Figure 1: Convergence of $\{x_m\}$ with different initial values for $n = 20$.

$A(x) = -x$, $W(x) = 2x$, $U(x) = x/2$, $V(x) = \sin(x)$, and $f(x) = x/3$, $x \in \mathbb{R}$, respectively. Here A is a bounded linear operator, W is a strongly positive bounded linear operator with coefficient $\zeta = 2$, and f is a τ-Lipschitzian mapping with coefficient $\tau = 1/3$. Also, both U and V are non-expansive mappings.

Let $E_1 : \mathbb{R} \to \mathbb{R}$ be defined as $E_1(x) = 2x$ and $E_2 : \mathbb{R} \to \mathbb{R}$ be defined as $E_2(x) = -4x/5$. For $\lambda = 1/4$, we compute the resolvent of E_1 and E_2 as

$$J_{\lambda,E_1}(x) = \frac{2}{3}x, \qquad \text{and} \qquad J_{\lambda,E_2}(x) = \frac{5}{4}x.$$

It can be easily seen that $\Omega = \{0\}$ here.

Also, define $G_1(z, y) = 3y^2 + 2zy - 5z^2$ and $G_2(z, y) = y^2 - z^2$, and put $\phi_1(z, y) = \phi_2(z, y) = 0$. It is easy to see that G_1 and G_2 satisfy conditions (i)–(iv) of Assumptions 1. Therefore, for $r_m = r > 0$, $T_r^{G_1}(x)$ is non-empty and single-valued for each $x \in C$. Hence, for $r > 0$, there exists $z \in C$ such that

$$G_1(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \qquad \text{for any } y \in C,$$

which is equivalent to

$$3ry^2 + (z - x + 2rz)y + (xz - 5rz^2 - z^2) \geq 0, \qquad y \in C.$$

After solving the above inequality, we get $z = x/(1+8r)$ for each $r > 0$, i.e., $T_r^{G_1}(x) = x/(1+8r)$ for each $r > 0$. Similarly, $T_r^{G_2}(x) = x/(1+2r)$. It can be easily seen that $\mathcal{S} = \{0\}$ here. This implies that $\mathcal{F} = \Omega \cap \mathcal{S} = \{0\}$. Now, let us put $r = 1/8$, $\gamma = 1$ and

$$\sigma_m = \frac{1}{m+6}, \quad b_m = \frac{m+1}{6(m+3)}, \quad \eta_m^1 = \frac{m+3}{m+4}, \quad \xi_m^1 = \frac{m+4}{m+5}$$

for each $m \geq 1$. Then $\rho\zeta > \gamma\tau$ and the sequences $\{\sigma_m\}$, $\{b_m\}$, $\{\eta_m^1\}$, and $\{\xi_m^1\}$ satisfy the conditions of Theorems 1, 2 and 3.

In Table 1, we present iterations of the Algorithm for different initial values, which are illustrated on Figure 1. It can be seen, that the constructed sequence $\{x_m\}$ converges to 0.

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