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The soft Jacobson radical of a commutative ring

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Abstract. In this paper, the notion of the soft Jacobson radical of a ring is defined. A relationship between the soft Jacobson radical of a ring and Jacobson semisimple ring is established. Some properties of this notion have been studied under homomorphism.

Key Words: Jacobson radical, Jacobson semisimple ring, soft maximal int-ideal, soft Jacobson radical
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Introduction

The idea of a soft set was introduced by D. Molodtsov [14] in 1999 as a parametrized mathematical tool which has many applications in medical sciences, engineering, economics, etc. Maji and Roy [12] presented a beautiful problem where soft sets are applied in decision making. Applications of soft sets in algebra have been studied rapidly in the last two decades.

Algebraic structure in soft set theory was introduced by Aktas and Cagman [2] in 2007. They defined a soft group as a parametrized family of subgroups of the given group. Extending the notion of a soft group, several algebraic structures like soft ring [1], soft ideal [6], soft vector space [15] etc., have been introduced. In 2012, Cagman et al. [5] defined group structure on a soft set in a new way using set inclusion relation. This concept has been named as soft int-group. Already many researchers have spent their times to extend the notion of soft int-group for further development of soft set theory in the same direction. As a result, the notions of soft int-ring [7], soft int-field, soft int-module [4], soft int-ideal [7] etc, have been established. The soft radical of a soft int-ideal is defined in [9].

This paper focuses on introducing the concept of the soft Jacobson radical of a ring. We define the soft Jacobson radical of a commutative ring R with unity as the soft intersection of all soft maximal int-ideals of R. A connection between the soft Jacobson radical of a ring and Jacobson semisimple ring has been studied. It is established that the homomorphic image of the soft Jacobson radical of a ring R is equal to the soft Jacobson radical of a homomorphic image of R under certain condition. If f is an epimorphism from a ring R to a ring R', then we prove that the homomorphic pre-image of the soft Jacobson radical of R' is equal to the soft Jacobson radical of R under a suitable condition.

1 Preliminaries

We include some basic definitions and results of soft set theory which will be useful in the next section. Throughout this paper, unless otherwise is stated, U is the initial universe, E is the set of parameters, $P(U)$ is the power set of U, and $A \subseteq E$.

Let F be a mapping given by $F : A \to P(A)$. A pair (F, A) is called a soft set of A over U. When no confusions regarding the parameter set A and the universal set U arise, the soft set (F, A) is simply denoted by F. The collection of all soft sets with parameter set A over U will be denoted by $S(A, U)$.

Let $F, G \in S(A, U)$. If $F(t) \subseteq G(t)$ for all $t \in A$, then F is called a soft subset of G and is denoted by $F \tilde{\subseteq} G$. Here G is called a soft superset of F and it is denoted by $G \tilde{\supseteq} F$. We write $F = G$, if $F(t) = G(t)$ for all $t \in A$.

Let $F \in S(A, U)$ and $K \subseteq U$, the set $F_K = \{x \in A : F(x) \supseteq K\}$ is called K-inclusion subset of the soft set F.

The soft intersection $F \tilde{\cap} G$ of two soft sets F and G is defined by

$$(F \tilde{\cap} G)(t) = F(t) \cap G(t), \qquad t \in A.$$

It is not difficult to see that the following statement holds true.

Proposition 1 *Let $F, G \in S(A, U)$ and $K \subseteq U$. Then $(F \tilde{\cap} G)_K = F_K \cap G_K$.*

Let A and A' be some parameter sets and let $f : A \to A'$ be any mapping. The soft image $f(F)$ of F under f is defined by

$$f(F)(y) = \begin{cases} \bigcup_{x \in f^{-1}(y)} F(x), & \text{if } f^{-1}(y) \neq \emptyset, \\ \emptyset, & \text{otherwise,} \end{cases}$$

for all $y \in A'$.

Let $A, A' \subseteq E$, $G \in S(A', U)$ and let $f : A \to A'$ be an onto mapping. The soft preimage of G under f is a soft set $f^{-1}(G) \in S(A, U)$ such that $f^{-1}(G)(s) = G(f(s))$ for all $s \in A$.

A soft set $F \in S(A, U)$ is called an f–invariant soft set if from $f(x) = f(y)$ it follows that $F(x) = F(y)$, $x, y \in A$.

Let R be a ring. A soft set $F \in S(R, U)$ is called a soft int-ring of R if
(i) $F(x - y) \supseteq F(x) \cap F(y)$ and
(ii) $F(xy) \supseteq F(x) \cap F(y)$ for all $x, y \in R$.

A soft set $F \in S(R, U)$ is called a soft int-ideal of R if
(i) $F(x - y) \supseteq F(x) \cap F(y)$ and
(ii) $F(xy) \supseteq F(x) \cup F(y)$ for all $x, y \in R$.

Proposition 2 *[7] Let F be a soft int-ring or soft int-ideal of a ring R. Then $F(0) \supseteq F(r)$ for all $r \in R$, where 0 is the zero element of R.*

A soft int-ideal F of a commutative ring R is called a soft maximal int-ideal if F_L is a maximal ideal of R for $L = F(0)$.

Let F be a soft int-ideal of a ring R and $a \in R$. The soft set F_a of R, defined by $F_a(r) = F(r - a)$, $r \in R$, is called the soft coset of F in R determined by $a \in R$.

We denote the set of all soft cosets of a soft int-ideal F of a ring R by $R \wr F$.

Theorem 1 *[9] Let F be a soft int-ideal of a ring R. Then $R \wr F$ forms a ring with respect to binary compositions $F_a + F_b = F_{a+b}$ and $F_a F_b = F_{ab}$ for all $a, b \in R$. Here F_0 is the zero element of $R \wr F$. If R is a ring with unity 1, then F_1 is the unity element in $R \wr F$.*

Proposition 3 *[9] Let F be a soft int-ideal of a ring R. Then $F_a = F_0$ if and only if $F(a) = F(0)$, where $a \in R$.*

Theorem 2 *[10] Let $f : R \to R'$ be an epimorphism, where R, R' are two commutative rings with unity. If F is an f–invariant soft maximal int-ideal of R, then $f(F)$ is a soft maximal int-ideal of R'.*

Theorem 3 *[10] Let $f : R \to R'$ be a homomorphism, where R, R' are two commutative rings with unity. If F' is a soft maximal int-ideal of R', then $f^{-1}(F')$ is an f–invariant soft maximal int-ideal of R.*

The following statement is easy to be verified.

Proposition 4 *Let $A, A' \subseteq E$ and $f : A \to A'$ be any onto mapping. If $G \in S(A', U)$, then $f(f^{-1}(G)) = G$.*

Proposition 5 *[5] Let $A, A' \subseteq E$ and $f : A \to A'$ be any mapping. Let $F_i \in S(A', U)$ for all $i \in \Delta$, where Δ is the non-empty index set. Then $f^{-1}\left(\widetilde{\bigcap_{i\in\Delta}} F_i\right) = \widetilde{\bigcap_{i\in\Delta}} f^{-1}(F_i)$.*

Proposition 6 *[9] Let f be any mapping from a set A to a set A'. If F is an f–invariant soft set of A, then $f^{-1}(f(F)) = F$.*

A ring R is called Jacobson semisimple ring if the Jacobson radical $JR(R) = \{0\}$.

Theorem 4 *[13] The quotient ring $R/JR(R)$ is a Jacobson semisimple ring.*

2 Soft Jacobson Radical of a Ring

Throughout this section, R is a commutative ring with unity 1. In ring theory [13], the Jacobson radical $JR(R)$ of the ring R is defined as the intersection of all maximal ideals of R. Here, we introduce the following notion.

The soft Jacobson radical $SJR(R)$ of the ring R is defined by

$$SJR(R) = \widetilde{\bigcap}\{F : F \text{ is a soft maximal int-ideal of } R\}.$$

Theorem 5 *[13] Let $y \in R$. Then $y \in JR(R)$ if and only if $1-xy$ is a unit in R for all $x \in R$.*

Theorem 6 *Let $y \in R$ and $F = SJR(R)$. Then $y \in F_K$ with $K = F(0)$ if and only if $1 - xy$ is a unit in R for all $x \in R$.*

Proof. Let $y \in F_K$ for $K = F(0)$. Suppose $1 - xy$ is not a unit in R for some $x \in R$. Then, by the crisp concept, there exists a maximal ideal I of R such that $1 - xy \in I$. Define a soft int-ideal H of R over U by

$$H(r) = \begin{cases} N, & \text{if } r \in I, \\ L, & \text{if } r \in R - I, \end{cases}$$

where $L \subset N \subseteq U$, $L \subset K$. Hence, H is a soft maximal int-ideal of R. Now, $y \in F_K$ implies $F(y) \supseteq K$. Since H is a soft maximal int-ideal of R and $F = SJR(R)$, we get $H(y) \supseteq F(y)$. Thus, $H(y) \supseteq K \supset L$. This implies $H(y) = N$, and hence, $y \in I$. Then $xy \in I$. Therefore, $1 = 1 - xy + xy \in I$. This implies that I is not a maximal ideal of R, which is a contradiction. Therefore, $1 - xy$ is a unit in R for all $x \in R$.

Conversely, assume that $1 - xy$ is a unit in R for all $x \in R$. Then by Theorem 5, $y \in JR(R)$. Hence, $y \in M$ for any maximal ideal M of R. Now, let $F = \widetilde{\bigcap}_i F_i$, where F_i is a soft maximal int-ideal of R for $1 \leq i \leq n$, $i, n \in \mathbb{N}$. Since $(F_i)_{L_i}$ (where $L_i = F_i(0)$) is a maximal ideal of $R, y \in (F_i)_{L_i}$ for $1 \leq i \leq n$. Again, $K = F(0) = \bigcap_i F_i(0)$. Then $L_i = F_i(0) \supseteq K$ for $1 \leq i \leq n$, $n \in \mathbb{N}$. Therefore, $y \in (F_i)_K$ for all i. By Proposition 1, $\left(\widetilde{\bigcap}_i F_i\right)_K = \bigcap_i (F_i)_K$. Hence, $y \in \left(\widetilde{\bigcap}_i F_i\right)_K = F_K$. $\square$

Theorem 7 *Let $F = SJR(R)$. Then $r \in R$ is a unit in R if and only if F_r is a unit in $R \wr F$.*

Proof. Let $r \in R$ be a unit in R. Then there exists $s \in R$ such that $rs = 1$. Hence, $F_{rs} = F_1$, where F_1 is the unity element in $R \wr F$ (by Theorem 1). This implies $F_r F_s = F_1$. Thus, F_r is a unit in $R \wr F$.

Conversely, let F_r be a unit in $R \wr F$ for $r \in R$. Then there exists $F_s \in R \wr F$ such that $F_r F_s = F_1$. This implies $F_{rs} = F_1$, and therefore, $F(x - rs) = F(x - 1)$ for all $x \in R$. Particularly, $F(1 - rs) = F(0)$.

Let I be any maximal ideal of R. Define a soft int-ideal H of R by

$$H(a) = \begin{cases} F(0), & \text{if } a \in I, \\ L, & \text{if } a \in R - I, \end{cases}$$

where $L \subset F(0)$. Then H is a soft maximal int-ideal of R and $H(0) = F(0)$. Since F is the soft Jacobson radical of R, $F(0) = F(1 - rs) \subseteq H(1 - rs)$. This implies $H(1-rs) = H(0)$, and hence, $1-rs \in I$. Thus, $1-rs \in JR(R)$, since I is any maximal ideal of R. Then by Theorem 5, $rs = 1 - (1 - rs)$ is a unit in R. Therefore, r is a unit in R. □

Theorem 8 *Let $F = SJR(R)$. Then the ring $R \wr F$ is a Jacobson semisimple ring.*

Proof. To prove that the ring $R \wr F$ is a Jacobson semisimple ring, we have to prove $JR(R \wr F) = \{F_0\}$. Let $F_r \in JR(R \wr F)$. Then by Theorem 5, $F_1 - F_x F_r$ is a unit in $R \wr F$ for all $F_x \in R \wr F$. There exists $F_y \in R \wr F$ such that $(F_1 - F_x F_r)F_y = F_1$. By Theorem 1, $F_{(1-xr)y} = F_1$. Hence, $F_{(1-xr)y}(a) = F_1(a)$ for all $a \in R$. Then $F(a - y + xry) = F(a - 1)$, $a \in R$. Therefore,

$$F(1 - y + xry) = F(0). \tag{1}$$

Let I be any maximal ideal of R. We define a soft int-ideal H of R by

$$H(a) = \begin{cases} F(0), & \text{if } a \in I, \\ L, & \text{if } a \in R - I, \end{cases}$$

where $L \subset F(0)$. Then H is a soft maximal int-ideal of R and $H(0) = F(0)$. Since F is the soft Jacobson radical of R, Equation (1) implies

$$H(1 - y + xry) \supseteq F(1 - y + xry) = F(0) = H(0).$$

By Proposition 2, $H(1 - y + xry) = H(0)$. Therefore, $1 - y + xry \in I$ for any maximal ideal I of R. This implies $1 - y + xry \in JR(R)$. Then by Theorem 5, $1 - (1 - y + xry) = y - xry$ is a unit in R. This implies that $1 - xr$ is a unit in R, and hence, $r \in JR(R)$.

Suppose $\{G_i : i \in \Gamma\}$ is a collection of all soft maximal int-ideals of R, where Γ is the index set. Then $(G_i)_{L_i}$ is the maximal ideal of R, where $L_i = G_i(0)$. From $r \in JR(R)$ it follows that $r \in (G_i)_{L_i}$. Hence, $G_i(r) = L_i = G_i(0)$, and therefore,

$$F(r) = \bigcap_{i\in\Gamma} G_i(r) = \bigcap_{i\in\Gamma} G_i(0) = F(0).$$

By Proposition 3, we have $F_r = F_0$. Thus, $JR(R \wr F) = \{F_0\}$. Therefore, $R \wr F$ is a Jacobson semisimple ring. □

Proposition 7 *Let $A, A' \subseteq E$ and $f : A \to A'$ be any onto mapping. Let $F_1, F_2 \in S(A, U)$. If F_1, F_2 are both f–invariant, then $f(F_1 \widetilde{\cap} F_2) = f(F_1)\widetilde{\cap} f(F_2)$.*

Proof. Since F_1, F_2 are both f–invariant soft sets, from $f(x) = f(y)$ it follows that $F_1(x) = F_1(y)$ and $F_2(x) = F_2(y)$, where $x, y \in A$. Hence, $(F_1\widetilde{\cap}F_2)(x) = F_1(x) \cap F_2(x) = F_1(y) \cap F_2(y) = (F_1\widetilde{\cap}F_2)(y)$. Therefore, $F_1\widetilde{\cap}F_2$ is also f–invariant soft set.

Now, for any $y \in A$, using the fact that $F_1\widetilde{\cap}F_2$ is f–invariant, we have

$$\begin{aligned} f(F_1\widetilde{\cap}F_2)(y) &= \bigcup_{z\in f^{-1}(y)} (F_1\widetilde{\cap}F_2)(z) \\ &= (F_1\widetilde{\cap}F_2)(y) \\ &= F_1(y) \cap F_2(y). \end{aligned}$$

Since F_1 and F_2 are f–invariant, further we can write

$$\begin{aligned} &[f(F_1)\widetilde{\cap}f(F_2)](y) \\ &= f(F_1)(y) \cap f(F_2)(y) \\ &= \left[\bigcup_{s\in f^{-1}(y)} F_1(s)\right] \bigcap \left[\bigcup_{t\in f^{-1}(y)} F_2(t)\right] \\ &= F_1(y) \cap F_2(y). \end{aligned}$$

Therefore, $f(F_1\widetilde{\cap}F_2) = f(F_1)\widetilde{\cap}f(F_2)$. □

The following proposition is a generalization of Proposition 7 for an arbitrary soft intersection.

Proposition 8 *Let $A, A' \subseteq E$ and $f : A \to A'$ be any onto mapping. Let $F_i \in S(A, U)$ be f–invariant for all $i \in \Gamma$, where Γ is an arbitrary non-empty index set. Then $f\left(\widetilde{\bigcap_{i\in\Gamma}} F_i\right) = \widetilde{\bigcap_{i\in\Gamma}} f(F_i)$.*

Theorem 9 *Let $f : R \to R'$ be a homomorphism, where R, R' are two commutative rings with unity. If each soft maximal int-ideal of R is f–invariant, then $f(SJR(R)) = SJR(f(R))$.*

Proof. Let $\{F_i : i \in \Gamma\}$ be the complete collection of soft maximal int-ideals of R, where Γ is an arbitrary non-empty index set. Then by the definition of soft Jacobson radical, we have $SJR(R) = \widetilde{\bigcap\limits_{i\in\Gamma}} F_i$. Let F_i be f–invariant for all $i \in \Gamma$. Then by Proposition 8, we have

$$f(SJR(R)) = f\left(\widetilde{\bigcap_{i\in\Gamma}} F_i\right) = \widetilde{\bigcap_{i\in\Gamma}} f(F_i).$$

Since each $F_i(i \in \Gamma)$ is an f-invariant soft maximal int-ideal of R, by Theorem 2, we have $f(F_i)$ is a soft maximal int-ideal of $f(R)$ for all $i \in \Gamma$.

Suppose G is a soft maximal int-ideal of $f(R)$. Then by Theorem 3, $f^{-1}(G)$ is an f–invariant soft maximal int-ideal of R. Hence, by Proposition 4, $f(f^{-1}(G)) = G$. Thus, $\{f(F_i) : i \in \Gamma\}$ is the complete collection of soft maximal int-ideals of $f(R)$. Hence, $\widetilde{\bigcap\limits_{i\in\Gamma}} f(F_i) = SJR(f(R))$. Therefore, $f(SJR(R)) = SJR(f(R))$. □

Theorem 10 *Let $f : R \to R'$ be an epimorphism, where R, R' are two commutative rings with unity. Then $f^{-1}(SJR(R'))\widetilde{\supseteq}SJR(R)$. Moreover, if each soft maximal int-ideal of R is f–invariant, then $f^{-1}(SJR(R')) = SJR(R)$.*

Proof. Let $\{G_j : j \in \Delta\}$ be the complete collection of all soft maximal int-ideals of R'. Hence, by the definition of soft Jacobson radical, we have $SJR(R') = \widetilde{\bigcap\limits_{j\in\Delta}} G_j$. By Proposition 5,

$$f^{-1}(SJR(R')) = f^{-1}\left(\widetilde{\bigcap_{j\in\Delta}} G_j\right) = \widetilde{\bigcap_{j\in\Delta}} f^{-1}(G_j).$$

By Theorem 3, $f^{-1}(G_j)$ is an f–invariant soft maximal int-ideal of R for all $j \in \Delta$. There may be some soft maximal int-ideal of R that are not f–invariant. Hence,

$$\widetilde{\bigcap_{j\in\Delta}} f^{-1}(G_j)\widetilde{\supseteq}SJR(R). \tag{2}$$

Therefore, $f^{-1}(SJR(R'))\widetilde{\supseteq}SJR(R)$.

Now, we assume that each soft maximal int-ideal of R is f–invariant. Let F be a soft maximal int-ideal of R. Then by Theorem 2, $f(F)$ is a soft maximal int-ideal R' and by the Proposition 6, $f^{-1}(f(F)) = F$. Hence, for each soft maximal int-ideal F of R, there exist soft maximal int-ideal $f(F)$ of R' such that $f^{-1}(f(F)) = F$. Thus, from Equation (2), we have $\widetilde{\bigcap\limits_{j\in\Delta}} f^{-1}(G_j) = SJR(R)$. Therefore, $f^{-1}(SJR(R')) = SJR(R)$. □

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