Armenian Journal of Mathematics
Volume 2, Number 4, 2009, 135–145

Some fixed point theorems for pointwise R-weakly commuting hybrid mappings in metrically convex spaces

Amit Singh*, R.C. Dimri** and Smita Joshi

Department of Mathematics, H.N.B. Garhwal University
Srinagar (Garhwal)-246174, India.

* *singhamit841@gmail.com*

** *dimrirc@gmail.com*

Abstract

In the present paper we prove some coincidence common fixed point theorems for a family of hybrid pairs of mappings in metrically convex spaces by using the notion of pointwise R-weakly commuting mappings.

Key Words: Fixed point, hybrid contractive condition, metrically convex metric spaces, R-weakly commuting mappings.
Mathematics Subject Classification 2000: Primary 54H25, 47H10.

1 Introduction

Fixed point theorems for single-valued and multivalued mappings have been studied extensively and applied to diverse problems during the last few decades. Nadler [17] introduced the concept of multivalued contraction mappings and established that a multivalued contraction mapping possesses a fixed point in a complete metric space. Subsequently, many authors have generalized Nadler's fixed point theorem in different ways. Assad and Kirk [4] gave sufficient conditions for non-self mappings to ensure the fixed point by proving a result on multivalued contractions in complete metrically convex metric spaces. Several authors proved some fixed point theorems for non-self mappings (see, for instance [1], [2], [11], [12], [13], [15], [19]).

Recently, Imdad and Khan [12] and Dhage, Dolhare and Petrusel [8] proved some fixed point theorems for a sequence of set-valued mappings which generalize several results due to Itoh [13], Khan [15], Ahmad and Imdad [1, 2], Ahmad and Khan [3] and others. The purpose of this paper is to prove some coincidence and common fixed point theorems for a sequence of hybrid type non-self mappings satisfying certain contraction condition by using R-weakly commutativity between multivalued mappings and single-valued mappings. Our results generalize and unify the results due to Imdad and Khan [12], Khan [15], Itoh [13], Ahmad and Imdad [1, 2], Ahmad and Khan [3] and several others.

2 Preliminaries

Let (X, d) be a metric space. Then following Nadler [17], we recall

(i) $CB(X) = \{$A: A is nonempty closed and bounded subset of X$\}$

(ii) $C(X) = \{$A: A is nonempty compact subset of X$\}$

(iii) For nonempty subsets A, B of X and $x \in X$, $d(x, A) = \inf\{d(x, a) : a \in A\}$,

$$H(A, B) = \max[\{supd(a, B) : a \in A\}, \{supd(A, b) : b \in B\}]. \tag{1}$$

It is well known that $CB(X)$ is a metric space with the distance H which is known as Hausdroff-Pompeiu metric on X.
The following definitions and lemmas will be frequently used in the sequel.

Definition 1 *[10]. Let K be a nonempty subset of a metric space (X, d), $T : K \to X$ and $F : K \to CB(X)$. The pair (F, T) is said to be pointwise R-weakly commuting on K if for given $x \in K$ and $Tx \in K$, there exists some $R = R(x) > 0$ such that*

$$d(Ty, FTx) \leq R.d(Tx, Fx) \tag{2}$$

for each $y \in K \bigcap Fx$. Moreover, the pair (F, T) will be called R-weakly commuting on K if (2) holds for each $x \in K$ and $Tx \in K$ with some $R > 0$.

If $R = 1$, we get the definition of weak commutativity of (F, T) on K. For $K = X$ definition 1 reduces to "Pointwise R-weakly commutativity" for single valued self mappings due to Pant [18].

Definition 2 *[9, 10]. Let K be a nonempty subset of a metric space $(X, d), T : K \to X$ and $F : K \to CB(X)$. The pair (F, T) is said to be weakly commuting if for every $x, y \in K$ with $x \in Fy$ and $Ty \in K$, we have*

$$d(Tx, FTy) = d(Ty, Fy). \tag{3}$$

Definition 3 *[10]. Let K be a nonempty subset of a metric space (X,d), $T:K\to X$ and $F:K\to CB(X)$. The pair (F,T) is said to be compatible if for every sequence $\{x_n\}\subset K$, from the relation*

$$\lim_{n\to\infty} d(Fx_n,Tx_n)=0 \tag{4}$$

and $Tx_n\in K$ (for every $n\in N$) it follows that $\lim_{n\to\infty} d(Ty_n,FTx_n)=0$, for every sequence $\{y_n\}\subset K$ such that $y_n\in Fx_n$, $n\in N$.

For hybrid pairs of self type mappings these definitions were introduced by Kaneko and Seesa [14].

Definition 4 *[11]. Let K be a nonempty subset of a metric space (X,d), $T:K\to X$ and $F:K\to CB(X)$. The pair (F,T) is said to be quasi-coincidentally commuting if for all coincidence points 'x' of (T,F), $TFx\subset FTx$ whenever $Fx\subset K$ and $Tx\in K$ for all $x\in K$.*

Definition 5 *[11]. A mapping $T:K\to X$ is said to be coincidentally idempotent w.r.t. mapping $F:K\to CB(X)$, if T is idempotent at the coincidence points of the pair (F,T).*

Definition 6 *[4]. A metric space (X,d) is said to be metrically convex if for any $x,y\in X$ with $x\neq y$ there exists a point $z\in X$, $x\neq z\neq y$ such that*

$$d(x,z)+d(z,y)=d(x,y). \tag{5}$$

Lemma 1 *[4]. Let K be a nonempty closed subset of a metrically convex metric space (X, d), if $x\in K$ and $y\notin K$ then there exists a point $z\in\delta K$ (the boundary of K) such that $d(x,z)+d(z,y)=d(x,y)$.*

Lemma 2 *[17]. Let $A,B\in CB(X)$ and $a\in A$, then for any positive number $q<1$ there exists $b=b(a)$ in B such that $q.d(a,b)=H(A,B)$.*

3 Main results

Theorem 1 *Let (X,d) be a complete metrically convex metric space and K is a nonempty closed subset of X. Let $\{F_n\}_{n=1}^{\infty}:K\to CB(X)$ and $S,T:K\to X$ satisfying*

(iv) $\delta K\subseteq SK\bigcap TK$, $F_i(K)\bigcap K\subseteq SK$, $F_j(K)\bigcap K\subseteq TK$

(v) $Tx\in\delta K\Rightarrow F_i(x)\subseteq K$, $Sx\in\delta K\Rightarrow F_j(x)\subseteq K$ and

$$H[F_i(x),F_j(y)]\leq ad(Tx,Sy)+b\max\{d(Tx,F_i(x)),d(Sy,F_j(y))\}$$
$$+c\max\{d(Tx,Sy),d(Tx,F_i(x)),d(Sy,F_j(y))\} \tag{6}$$

where $i=2n-1$, $j=2n$, $(n\in N)$, $i\neq j$ for all $x,y\in K$ with $x\neq y,a,b\geq 0$ and $\{(a+2b+2c)+(a^2+ab+ac)/q\}<q<1$, *(vi)* (F_i, T) *and* (F_j, S) *are pointwise R-weakly commuting pairs,*

(vii) $\{F_n\}$, S *and* T *are continuous on* K.

Then (F_i, T) *and* (F_j, S) *have a point of coincidence.*

Proof. Firstly, we proceed to construct two sequences $\{x_n\}$ and $\{y_n\}$ in the following way:

Let $x \in \delta K$. Since $\delta K \subseteq TK$ there exists a point $x_0 \in K$ such that $x = Tx_0$. From the implication $Tx_0 \in \delta K$ which implies $F_1(x_0) \subseteq F_1(K) \bigcap K \subseteq SK$. Let $x_1 \in K$ be such that $y_1 = Sx_1 \in F_1(x_0) \subseteq K$. Since $y_1 \in F_1(x_0)$ there exists a point $y_2 \in F_2(x_1)$ such that

$$q.d(y_1, y_2) \leq H[F_1(x_0), F_2(x_1)] \tag{7}$$

Suppose $y_2 \in K$. Then $y_2 \in F_2(K) \bigcap K \subseteq TK$ implies that there exists a point $x_2 \in K$ such that $y_2 \in Tx_2$. Otherwise, if $y_2 \notin K$, then there exists a point $p \in \delta K$ such that

$$d(Sx_1, p) + d(p, y_2) = d(Sx_1, y_2)$$

Since $p \in \delta K \subseteq TK$, there exists a point $x_2 \in K$ with $p = Tx_2$ so that

$$d(Sx_1, Tx_2) + d(Tx_2, y_2) = d(Sx_1, y_2)$$

Let $y_3 \in F_3(x_2)$ be such that

$$q.d(y_2, y_3) \leq H[F_2(x_1), F_3(x_2)]$$

Thus on repeating the foregoing arguments, we obtain two sequences $\{x_n\}$ and $\{y_n\}$ such that

(viii) $y_{2n} \in F_{2n}(x_{2n-1}), y_{2n+1} \in F_{2n+1}(x_{2n})$,

(ix) $y_{2n} \in K \Rightarrow y_{2n} = Tx_{2n}$ or $y_{2n} \notin K \Rightarrow Tx_{2n} \in \delta K$ and

$$d(Sx_{2n-1}, Tx_{2n}) + d(Tx_{2n}, y_{2n}) = d(Sx_{2n-1}, y_{2n})$$

(x) $y_{2n+1} \in K \Rightarrow y_{2n+1} = Sx_{2n+1}$ or $y_{2n+1} \notin K \Rightarrow Sx_{2n+1} \in \delta K$ and

$$d(Tx_{2n}, Sx_{2n+1}) + d(Sx_{2n+1}, y_{2n+1}) = d(Tx_{2n}, y_{2n+1})$$

We denote

$$\left.\begin{array}{l} P_0 = \{Tx_{2i} \in \{Tx_{2n}\} : Tx_{2i} = y_{2i}, \\ P_1 = \{Tx_{2i} \in \{Tx_{2n}\} : Tx_{2i} \neq y_{2i}, \\ Q_0 = \{Sx_{2i+1} \in \{Sx_{2n+1}\} : Sx_{2i+1} = y_{2i+1}, \\ Q_1 = \{Sx_{2i+1} \in \{Sx_{2n+1}\} : Sx_{2i+1} \neq y_{2i+1}. \end{array}\right\}$$

First we show that $(Tx_{2n}, Sx_{2n+1}) \notin P_1 \times Q_1$ and $(Sx_{2n-1}, Tx_{2n}) \notin Q_1 \times P_1$. If $Tx_{2n} \in P_1$, then $y_{2n} \neq Tx_{2n}$ and we have $Tx_{2n} \in \delta K$ which implies that $y_{2n+1} \in F_{2n+1}(x_{2n}) \subseteq K$. Hence $y_{2n+1} = Sx_{2n+1} \in Q_0$. Similarly, one can argue that $(Sx_{2n-1}, Tx_{2n}) \notin Q_1 \times P_1$.

Now we distinguish the following three cases:

Case 1. If $(Tx_{2n}, Sx_{2n+1}) \in P_0 \times Q_0$, then

$$\begin{aligned} q.d(Tx_{2n}, Sx_{2n+1}) &\leq H[F_{2n+1}(x_{2n}), F_{2n}(x_{2n-1})] \\ &\leq ad(Tx_{2n}, Sx_{2n-1}) + b\max\{d(Tx_{2n}, F_{2n+1}(x_{2n})), d(Sx_{2n-1}, F_{2n}(x_{2n-1}))\} \\ &\quad + c\max\{d(Tx_{2n}, Sx_{2n+1}), d(Tx_{2n}, F_{2n+1}(x_{2n})), d(Tx_{2n}, F_{2n+1}(x_{2n}))\} \\ &\leq ad(y_{2n}, y_{2n-1}) + b\max\{d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\} \\ &\quad + c\max\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\} \end{aligned}$$

which in turn yields

$$d(Tx_{2n}, Sx_{2n+1}) \leq \begin{cases} \left(\frac{a+b+c}{q}\right) d(Sx_{2n-1}, Tx_{2n}), if d(y_{2n-1}, y_{2n}) \geq d(y_{2n+1}, y_{2n}) \\ \left(\frac{a}{q-b-c}\right) d(Sx_{2n-1}, Tx_{2n}), if d(y_{2n-1}, y_{2n}) \leq d(y_{2n+1}, y_{2n}), \end{cases}$$

or

$$d(Tx_{2n}, Sx_{2n+1}) \leq hd(Sx_{2n-1}, Tx_{2n}),$$

where $h = max\{((a+b+c)/q), (a/(q-b-c))\} < 1$, since $\{(a+2b+2c)+(a^2+ab+ac)/q\} < 1$. Similarly if $(Sx_{2n-1}, Tx_{2n}) \in Q_0 \times P_0$, then

$$d(Sx_{2n-1}, Tx_{2n}) \leq \begin{cases} \left(\frac{a+b+c}{q}\right) d(Sx_{2n-1}, Tx_{2n-2}), if d(y_{2n-2}, y_{2n-1}) \geq d(y_{2n-1}, y_{2n}) \\ \left(\frac{a}{q-b-c}\right) d(Sx_{2n-1}, Tx_{2n-2}), if d(y_{2n-2}, y_{2n-1}) \leq d(y_{2n-1}, y_{2n}), \end{cases}$$

or

$$d(Sx_{2n-1}, Tx_{2n}) \leq h.d(Sx_{2n-1}, Tx_{2n-2}),$$

where $h = max\{((a+b+c)/q), (a/(q-b-c))\} < 1$, since $\{(a+2b+2c)+(a^2+ab+ac)/q\} < 1$.

Case 2. If $(Tx_{2n}, Sx_{2n+1}) \in P_0 \times Q_1$, then

$$d(Tx_{2n}, Sx_{2n+1}) + d(Sx_{2n+1}, y_{2n+1}) = d(Tx_{2n}, y_{2n+1})$$

which in turn yields

$$d(Tx_{2n}, Sx_{2n+1}) \leq d(Tx_{2n}, y_{2n+1}) = d(y_{2n}, y_{2n+1})$$

and hence

$$q.d(Tx_{2n}, Sx_{2n+1}) \leq q.d(y_{2n}, y_{2n+1})$$

$$\leq H[F_{2n+1}(x_{2n}), F_{2n}(x_{2n-1})].$$

Now proceeding as in case 1, we have

$$d(Tx_{2n}, Sx_{2n+1}) \leq \begin{cases} \left(\frac{a+b+c}{q}\right) d(Sx_{2n-1}, Tx_{2n}), if d(y_{2n-1}, y_{2n}) \geq d(y_{2n+1}, y_{2n}) \\ \left(\frac{a}{q-b-c}\right) d(Sx_{2n-1}, Tx_{2n}), if d(y_{2n-1}, y_{2n}) \leq d(y_{2n+1}, y_{2n}), \end{cases}$$

or

$$d(Tx_{2n}, Sx_{2n+1}) \leq hd(Sx_{2n-1}, Tx_{2n}),$$

where $h = max\{((a+b+c)/q), (a/(q-b-c))\} < 1$, since {(a + 2b + 2c) + (a^2 + ab + ac)/q} < 1. Similarly if $(Sx_{2n-1}, Tx_{2n}) \in Q_1 \times P_0$, then

$$d(Sx_{2n-1}, Tx_{2n}) \leq \begin{cases} \left(\frac{a+b+c}{q}\right) d(Sx_{2n-1}, Tx_{2n-2}), if d(y_{2n-2}, y_{2n-1}) \geq d(y_{2n-1}, y_{2n}) \\ \left(\frac{a}{q-b-c}\right) d(Sx_{2n-1}, Tx_{2n-2}), if d(y_{2n-2}, y_{2n-1}) \leq d(y_{2n-1}, y_{2n}), \end{cases}$$

or

$$d(Sx_{2n-1}, Tx_{2n}) \leq h.d(Sx_{2n-1}, Tx_{2n-2}),$$

where h = max {((a + b +c)/q), (a/(q – b – c))} < 1, since $\{(a+2b+2c)+(a^2+ab+ac)/q\} < 1$.

Case 3. If $(Tx_{2n}, Sx_{2n+1}) \in P_1 \times Q_0$, then $Sx_{2n-1} = y_{2n-1}$. Now proceeding as in case 1, one gets

$$\begin{aligned} q.d(Tx_{2n}, Sx_{2n+1}) = q.d(Tx_{2n}, y_{2n+1}) &\leq q.d(Tx_{2n}, y_{2n}) + q.d(y_{2n}, y_{2n+1}) \\ &\leq q.d(Sx_{2n-1}, y_{2n}) + H[F_{2n+1}(x_{2n}), F_{2n}(x_{2n-1})] \\ &\leq q.d(Sx_{2n-1}, y_{2n}) + ad(y_{2n}, y_{2n-1}) + b \max\{d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\} \\ &\quad + c \max\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1})\}, \end{aligned}$$

which in turn yields

$$d(Tx_{2n}, Sx_{2n+1}) \leq \begin{cases} \left(\frac{q+a}{q-b-c}\right) d(Sx_{2n-1}, Tx_{2n}), if d(y_{2n-1}, y_{2n}) \leq d(y_{2n+1}, y_{2n}) \\ \left(\frac{q+a+b+c}{q}\right) d(Sx_{2n-1}, Tx_{2n}), if d(y_{2n-1}, y_{2n}) \geq d(y_{2n+1}, y_{2n}.). \end{cases}$$

Now proceeding as earlier, one also obtain

$$d(Sx_{2n-1}, Tx_{2n}) \leq \begin{cases} \left(\frac{a+b+c}{q}\right) d(Sx_{2n-1}, Tx_{2n-2}), if d(y_{2n-2}, y_{2n-1}) \geq d(y_{2n-1}, y_{2n}) \\ \left(\frac{a}{q-b-c}\right) d(Sx_{2n-1}, Tx_{2n-2}), if d(y_{2n-2}, y_{2n-1}) \leq d(y_{2n-1}, y_{2n}). \end{cases}$$

Therefore combining above inequalities, we have

$$d(Tx_{2n}, Sx_{2n+1}) \leq k.d(Sx_{2n-1}, Tx_{2n-2})$$

where

$$k = \max \left\{ \begin{matrix} \left(\frac{a+b+c}{q}\right)\left(\frac{q+a}{q-b-c}\right), \left(\frac{a+b+c}{q}\right)\left(\frac{q+a+b+c}{q}\right), \\ \left(\frac{a}{q-b-c}\right)\left(\frac{q+a}{q-b-c}\right), \left(\frac{a}{q-b-c}\right)\left(\frac{q+a+b+c}{q}\right) \end{matrix} \right\} < 1,$$

since $\{(a+2b+2c)+(a^2+ab+ac)/q\} < 1$.

To substantiate that, the inequality $\{(a+2b+2c)+(a^2+ab+ac)/q\} < 1$ implies all foregoing inequalities, one may note that

$\{(a+2b+2c)+(a^2+ab+ac)/q\} < q \Rightarrow \{(aq+2bq+2cq)+(a^2+ab+ac)\} < q^2$,

$$aq\ + a^2 + bq + ab + cq + ac + bq + cq < q^2,$$

or

$$aq\ + a^2 + bq + ab + cq + ac < q^2 - bq - cq,$$

or

$$\left(\frac{a+b+c}{q}\right)\left(\frac{q+a}{q-\ b\ -\ c}\right) < 1$$

and

$$\{(a+2b+2c)+(a^2+ab+ac)/q\} < q \Rightarrow \{(a+b+c)+(a^2+ab+ac)/q\} < q$$

or

$$\{(aq+bq+cq)+(a^2+ab+ac)\} < q^2,$$

or

$$aq\ + a^2 + ab + ac + bq + cq < q^2,$$

or

$$aq\ + a^2 + ab + ac < q^2 - bq - cq$$

or

$$\left(\frac{a}{q\ -\ b-c}\right)\left(\frac{q+a+b+c}{q\ -\ a}\right) < 1.$$

Similarly one can establish the other inequalities as well. Thus in all the cases we have

$$d(Tx_{2n}, Sx_{2n+1}) \le k \max\{d(Sx_{2n-1}, Tx_{2n}), d(Tx_{2n-2}, Sx_{2n-1})\}$$

whereas

$$d(Sx_{2n+1}, Tx_{2n+1}) \le k \max\{d(Sx_{2n-1}, Tx_{2n}), d(Tx_{2n}, Sx_{2n+1})\}$$

Now on the lines of Assad and Kirk [4], it can be shown by induction that for n = 1, we have

$$d(Tx_{2n}, Sx_{2n+1}) \le k^n.\delta, d(Sx_{2n+1}, Tx_{2n+2}) \le k^{n+\frac{1}{2}}.\delta$$

Whereas

$$\delta = k^{-\frac{1}{2}} \max\{d(Tx_0, Sx_1), d(Sx_1, Tx_2)\}$$

Thus the sequence $\{Tx_0, Sx_1, Tx_2, Sx_3, ...Tx_{2n}, Sx_{2n+1}\}$ is a Cauchy sequence and hence converges to a point z in X. Now we assume that there exists a subsequence $\{Tx_{2n_k}\}$ of $\{Tx_{2n}\}$ which is contained in P_0. Further subsequences $\{Tx_{2n_k}\}$ and $\{Sx_{2n_k+1}\}$ both converge to $z \in K$ as K is closed subset of the complete metric space (X, d). Since $Tx_{2n_k} \in F_j(x_{2n_k-1})$

for any even integer $j \in N$ and $Sx_{2n_k-1} \in K$. Using pointwise R-weak commutativity of (F_j, S), we have

$$d(SF_j(x_{2n_k-1})), F_j(Sx_{2n_k-1})) \leq R_1.d(F_j(x_{2n_k-1})), Sx_{2n_k-1})) \tag{8}$$

for every even integer $j \in N$ with some $R_1 > 0$. Also

$$d(SF_j(x_{2n_k-1})), F_j(z)) \leq d(SF_j(x_{2n_k-1})), F_j(Sx_{2n_k-1})) + H(F_j(x_{2n_k-1})), F_j(z)). \tag{9}$$

Making $k \to \infty$ in (8) and (9) and using the continuity of S and F_j, we get $d(Sz, F_j(z)) \leq 0$ yielding thereby $Sz \in F_j(z)$, for any even integer $j \in N$.
Since $y_{2n_k+1} \in F_i(x_{2n_k})$ and $Tx_{2n_k} \in K$ for any odd integer $i \in N$. Using pointwise R-weak commutativity of (F_i, T), we have

$$d(TF_i(x_{2n_k})), F_i(Tx_{2n_k}) \leq R_2.d(F_i(x_{2n_k})), Tx_{2n_k})$$

for every odd integer $i \in N$with some $R_2 > 0$, besides

$$d(TF_i(x_{2n_k})), F_i(z)) \leq d(TF_i(x_{2n_k})), F_i(Tx_{2n_k})) + H(F_i(x_{2n_k})), F_i(z)).$$

Therefore as earlier the continuity of F_i and T implies $d(Tz, F_i(z)) \leq 0$ yielding thereby $Tz \in F_i(z)$, for any odd integer $i \in N$ as $k \to \infty$.
If we assume that there exists a subsequence $\{Sx_{2n_k+1}\}$ contained in Q_0, then analogous arguments establish the earlier conclusions. This concludes the proof.

Remark 1 *If we replace condition* (6) *by the condition*

$$H[F_i(x), F_j(y)] \leq a \max\{\tfrac{1}{2}d(Tx, Sy), d(Tx, F_i(x)), d(Sy, F_j(y))\}$$
$$+ b\{d(Tx, F_j(y)) + d(Sy, F_i(x))\}$$

then we get Theorem 3.4 [12].

Remark 2 *If we replace condition* (6) *by the condition*

$$H[F_i(x), F_j(y)] \leq a \max\{\tfrac{1}{2}d(Tx, Sy), d(Tx, F_i(x)), d(Sy, F_j(y))\}$$

$$+ b\{d(Tx, F_j(y)) + d(Sy, F_i(x))\}$$

and pointwise R-weakly commuting maps by compatible maps, then we get Theorem 3.1 due to Imdad and Khan [12].

Theorem 2 *Let (X, d) be a complete metrically convex metric space and K is a nonempty closed subset of X. Let* $\{F_n\}_{n=1}^{\infty} : K \to CB(X)$ *and S, T: K→ X satisfying* (6)*, (iv) and (v). Suppose that*
(xi) TK and SK are closed subspaces of X. Then

(*) (F_i, T) has a point of coincidence,

(**) (F_j, S) has a point of coincidence.

Moreover, (F_i, T) has a common fixed point if T is quasi-coincidentally commuting and coincidentally idempotent w.r.t. F_i whereas (F_j, S) has a common fixed point provided S is quasi-coincidentally commuting and coincidentally idempotent w.r.t. F_j.

Proof. On the lines of Theorem 1, one assumes that there exists a subsequence $\{Tx_{2n_k}\}$ which is contained in P_0 and TK as well as SK are closed subspaces of X. Since $\{Tx_{2n_k}\}$ is Cauchy in TK, it converges to a point $u \in TK$. Let $v \in T^{-1}u$, then $Tv = u$. Since $\{Sx_{2n_k+1}\}$ is a subsequence of Cauchy sequence, $\{Sx_{2n_k+1}\}$ converges to u as well. Using (6), one can write

$$\begin{aligned} q.d(F_i(v), Tx_{2n_k}) &\leq H[F_i(v), F_j(x_{2n_k-1})] \\ &\leq ad(Tv, Sx_{2n_k-1}) + b\max\{d(Tv, F_i(v)), d(Sx_{2n_k-1}, F_j(x_{2n_k-1}))\} \\ &\quad + c\max\{d(Tv, Sx_{2n_k-1}), d(Tv, F_i(v)), d(Sx_{2n_k-1}, F_j(x_{2n_k-1}))\} \end{aligned}$$

which on letting $k \to \infty$, reduces to

$$\begin{aligned} q.d(F_i(v), u) &\leq a(0) + b\max\{d(u, F_i(v)), 0\} + c\max\{0, d(u, F_i(v)), 0\} \\ &\leq (b+c).d(u, F_i(v)), \end{aligned}$$

yielding thereby $u \in F_i(v)$ which implies that $u = Tv \in F_i(v)$ as $F_i(v)$ is closed.

Since Cauchy sequence $\{Tx_{2n}\}$ converges to $u \in K$ and $u \in F_i(v)$, $u \in F_i(K) \bigcap K \subseteq SK$, there exists $w \in K$ such that Sw = u. Again using (6), one gets

$$\begin{aligned} q.d(Sw, F_j(w)) = q.d(Tv, F_j(w)) &\leq H[F_i(v), F_j(x_{2n_k-1})] \\ &\leq ad(Tv, Sw) + b\max\{d(Tv, F_i(v)), d(Sw, F_j(w))\} \\ &\quad + c\max\{d(Tv, Sw), d(Tv, F_i(v)), d(Sw, F_j(w))\} \leq (b+c).d(Sw, F_j(w)) \end{aligned}$$

implying thereby $Sw \in F_j(w)$, that is w is a coincidence point of (S, F_j).

If one assumes that there exists a subsequence $\{Sx_{2n_k+1}\}$ contained in Q_0 with TK as well as SK are closed subspaces of X, then noting that $\{Sx_{2n_k+1}\}$ is Cauchy in SK, the foregoing arguments establish that $Tv \in F_i(v)$ and $Sw \in F_j(w)$.

Since v is a coincidence point of (F_i, T) therefore using quasi-coincidentally commuting property of (F_i, T) and coincidentally idempotent property of T w.r.t. F_i, one can have

$$Tv \in F_i(v), u = Tv \Rightarrow Tu = TTv = Tv = u,$$

therefore $u = Tu = TTv \in TF_i(v) \subset F_i(Tv) = F_i(u)$ which shows that u is a common fixed point of (F_i, T). Similarly using the quasi-coincidentally commuting property of (F_j, S) and coincidentally idempotent property of S w.r.t. F_j, one can show that (F_j, S) has a common fixed point as well.

References

[1] A. Ahmad and M. Imdad. On common fixed point of mappings and multivalued mappings. Rad. Mat., 8(1), 1992, pp. 147-158.

[2] A. Ahmad and M. Imdad. Some common fixed point theorems for mappings and multivalued mappings. J. Math. Anal. Appl., 218(2), 1998, pp. 546-560.

[3] A. Ahmad and A.R. Khan. Some common fixed point theorems foe non-self hybrid contractions. J. Math. Anal. Appl., 213(1), 1997, pp. 275-286.

[4] N.A. Assad and W.A. Kirk. Fixed point theorems for set-valued mappings of contractive type. Pacific J. Math., 43(3), 1972, pp. 553-562.

[5] Lj. B. Ciric. On some non-expansive type mappings and fixed points. Indian J. Pure Appl. Math., 24(3), 1993, pp. 145-149.

[6] Lj. B. Ciric. Coincidence and fixed points of non-expansive type multivalued and single-valued maps. Indian J. Pure Appl. Math., 26(5), 1995, pp. 393-401.

[7] Lj. B. Ciric and J.S. Ume. On an extension of a theorem of Rhoades. Rev. Roumaine Math. Pures Appl., 49(2), 1995, pp. 103-112.

[8] B.C. Dhage, U.P. Dolhare and A. Petrusel. Some common fixed point theorems for sequence of nonself multivalued operators in metrically convex metric spaces. Fixed Point Theorey, 4(2), 2003, pp. 143-158.

[9] O. Hadzic. On coincidence points in convex metric spaces. Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat., 19(2), 1986, pp. 233-240.

[10] O. Hadzic and Lj. Gajic. Coincidence points for set-valued mappings in convex metric spaces. Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat., 16(1), 1986, pp. 13-25.

[11] M. Imdad, A. Amhad and S. Kumar. On nonlinear nonself hybrid contractions. Rad. Mat., 10(2), 2001, 233-244.

[12] M. Imdad and Ladlay Khan. Fixed point theorems for a family of hybrid pairs of mappings in metrically convex spaces. Fixed Point Theory and Applications, 3, 2005, pp. 281-294.

[13] S. Itoh. Multivalued generalized contractions and fixed point theorems. Comment. Math. Univ. Caroline, 18(2), 1977, pp. 247-258.

[14] H. Kaneko and S. Seesa. Fixed point theorems for compatible multivalued and single-valued mappings. Int. J. Math. Math. Sci., 12(2), 1989, pp. 257-262.

[15] M.S. Khan. Common fixed point theorems for multivalued mappings. Pacific J. Math., 95(2), 1981, pp. 337-347.

[16] M.S. Khan, H.K. Pathak and M.D. Khan. Some fixed point theorems in metrically convex spaces. Georgian Math. J., 7(3), 2000, pp. 523-530.

[17] S.B. Hadler Jr. Multi-valued contraction mappings. Pacific J. Math. 30(2), 1969, pp. 475-488.

[18] R.P. Pant. Common fixed points of non-commuting mappings. J. Math. Anal. Appl., 188(2), 1994, pp. 436-440.

[19] V. Popa. Coincidence and fixed point theorems for non-continuous hybrid contractions. Nonlinear Anal. Forum , 7(2), 2002, pp. 153-158.