ARMENIAN JOURNAL OF MATHEMATICS
Volume 3, Number 1, 2010, 1–13

An example on asymptotic on manifolds with corners

Gohar Harutyunyan*

* *Institut für Mathematik*
Carl von Ossietzky Universität Oldenburg
26111 Oldenburg
harutyunyan@mathematik.uni-oldenburg.de

Received by the editors April 29, 2009; accepted for publication January 13, 2010.

Abstract

In this paper we consider an example on asymptotic expansions on manifolds with corner singularities. The content is a helpful step to understand how affects the push–forward operation under a b-fibration on the asymptotic data on basic objects, namely on b-densities.

Key Words: Pseudodifferential calcul, corner singularity, push-forward, asymptotic expansion.
Mathematics Subject Classification 2000: 35S05, 35J70, 35C20

Introduction

Pseudodifferential calculus on configurations with (geometric) singularities, in particular, on manifolds with corners, is an important area of mathematical research. Further, asymptotic structure at the singular points is one of the main problems of singular analysis.

Any operation in analysis consists of some basic processes, such as pull–back and push–forward. In this view it is important to understand how affects the push-forward under an appropriate map on the asymptotic data of the elements under the consideration.

Let W, V be two manifolds with corner singularities and $f : W \to V$ be a b-fibration. Further, let μ be a compactly supported b-density on W. The Melrose's Push-Forward Theorem, see in [2], [4], or [5], shows that $f_*\mu$ is a b-density on V and defines its index family by that of μ and geometric properties of f.

The next natural point is to get the explicit asymptotic expansion of $f_*\mu$, i.e., to define the coefficients. For that purpose some general information is to find in [4].

In this paper we consider a special example, namely, the b-fibration $f(x, y) = xy$ between the manifolds $\overline{\mathbb{R}}_+^2$ and $\overline{\mathbb{R}}_+$. We get the asymptotic expansion of $f_*\mu$ for a compactly supported density μ on $\overline{\mathbb{R}}_+^2$ at 0. Note that this result can be find using the Singular Asymptotics Lemma by Brüning and Seeley, see in [1], or [3]. An advantage of our approach is that the reader is not assumed to be familar with the singular analysis, and this example can be helpful to understand more general results.

I wish to express my thanks to Prof. D. Grieser (University of Oldenburg) for valuable discussions and remarks.

1 Push–Forward Theorem

1.1 Spaces of polyhomogeneous conormal functions; b-integral

In this section we turn to necessary material with some important remarks.

Definition 1.1. *A discrete set*

$$E := \{(z_j, k_j)\}_{j\in\mathbb{N}}$$

with $(z_j, k_j) \in \mathbb{C} \times \mathbb{N}_0$ *is said to be an index set, if*

$$\operatorname{Re} z_j \to \infty \quad \textit{as} \quad j \to \infty \qquad \textit{and} \qquad (z, k) \in E \Rightarrow (z, s) \in E \quad \textit{for any} \quad s \leq k.$$

Definition 1.2. (i) *A boundary defining function on a manifold X with boundary ∂X is an up to the boundary smooth function $\rho : X \to [0, \infty)$ such that $\rho^{-1}(0) = \partial X$ and $d\rho \neq 0$ on ∂X.* (*A function ρ on a manifold with boundary is called smooth up to the boundary iff all derivatives of all orders of ρ are bounded on bounded subsets of X°.*)

(ii) *A trivialization near the boundary is a diffeomorphism $(x, y) : U \to [0, \varepsilon) \times \partial X$ where U is a neighbourhood of ∂X in X; in particular, x must then be a boundary defining function.*

By the next definition we introduce the basic function spaces used in this paper.

Definition 1.3. (i) *Let X be a n-dimensional manifold with boundary. The space $\Omega^E_{\mathrm{pc}}(X)$ of polyhomogeneous conormal functions on X with index set E is defined to be the set of all $u \in C^\infty(X^\circ)$ which, in some trivialization (x, y) near the boundary, satisfy*

$$\Big|(x\partial x)^\alpha \partial_y^\beta \Big(u(x, y) - \sum_{\substack{(z,k)\in E\\ \operatorname{Re} z\leq N}} a_{z,k}(y) x^z \log^k x\Big)\Big| \leq Cx^N \tag{1}$$

for every $N \in \mathbb{R}, \alpha \in \mathbb{N}_0, \beta \in \mathbb{N}_0^{n-1}$, *for some functions* $a_{z,k}(y) \in C^\infty(\partial X)$ *and some constant* $C = C(N, \alpha, \beta) > 0$, *uniformly on compact subsets of* ∂X.
If u *satisfies the above conditions we also write*

$$u(x,y) \sim \sum_{(z,k)\in E} a_{z,k}(y) x^z \log^k x \quad \textit{as} \quad x \to 0 \tag{2}$$

and call the sum on the right side of (2) *the asymptotic sum of* u *as* $x \to 0$.

(ii) *Let* X *be a manifold with corners and boundary hypersurfaces* $H_1, \dots, H_m$, *and let* $E_1, \dots, E_m$ *be index sets. We think of* E_i *as associated to* H_i. *The set* $\mathcal{E} = (E_1, \dots, E_m)$ *is called an index family for* X.

(iii) *Let us now define the corresponding function spaces on manifolds with corners. For the notational convenience we restrict to* $\overline{\mathbb{R}}_+^2$; *the extension to the general case does not cause any essential difficulty.*

The space $\Omega^{\mathcal{E}}_{\mathrm{pc}}(\overline{\mathbb{R}}_+^2)$ *of polyhomogeneous conormal functions on* $\overline{\mathbb{R}}_+^2$ *with index family* $\mathcal{E} = (E, F)$ *is defined to be the set of all functions* u *on* $\overline{\mathbb{R}}_+^2$ *that satisfy the conditions of* (i) *with* $Cy^{-J-1}x^N$ $(J = \inf F := \inf\{\mathrm{Re}\, w : \exists s \in N_0 \quad \textit{such that} \quad (w,s) \in F\})$ *as the right side of the inequality* (1) *and the coefficients* $a_{z,k}(y)$ *polyhomogeneous conormal with index set* F *in the sense of* (i) (*for the manifold with boundary* $\overline{\mathbb{R}}_+$).

Remark 1.4. *The definitions of the spaces* $\Omega^E_{pc}(X), \Omega^{\mathcal{E}}_{pc}(\overline{\mathbb{R}}_+^2)$, *etc. are correct, i.e., independent of the choice of boundary defining functions, if one assume that* E (*analogously* F) *satisfies the condition:*

$$(z,k) \in E \Rightarrow (z+1, k) \in E. \tag{3}$$

Definition 1.5. (i) *A* b*-density on* $\overline{\mathbb{R}}_+$ $(\overline{\mathbb{R}}_+^2)$ *polyhomogeneous conormal with index set* E (*index family* $\mathcal{E}$) *is a density of the form*

$$u(x)\frac{dx}{x} \qquad \left(u(x,y)\frac{dx}{x}\frac{dy}{y}\right)$$

with $u(x) \in \Omega^E_{\mathrm{pc}}(\overline{\mathbb{R}}_+)$ $(u(x,y) \in \Omega^{\mathcal{E}}_{\mathrm{pc}}(\overline{\mathbb{R}}_+^2))$.

Analogously, a b*-density on* $\overline{\mathbb{R}}_+^k \times \mathbb{R}^{n-k}$ *polyhomogeneous conormal with index family* $\mathcal{E} = (E_1, \dots, E_k)$ *is a density of the form*

$$u(x_1, \dots, x_n)\frac{dx_1}{x_1} \dots \frac{dx_k}{x_k} \cdot dy_1 \cdots dy_{n-k} \tag{4}$$

with $u(x_1, \dots, x_n) \in \Omega^{\mathcal{E}}_{\mathrm{pc}}(\overline{\mathbb{R}}_+^k \times \mathbb{R}^{n-k})$.
((4) *is independent of the choice of coordinates if each of* $E_i, i = 1, \dots, k$, *satisfies the condition* (3).)

(ii) *By* ${}^{\mathrm{b}}\Omega^{\mathcal{E}}_{\mathrm{pc}}(X)$ *we denote the set of all* b*-densities on the manifold with corner* X *polyhomogeneous conormal with index family* $\mathcal{E}$.

Remark 1.6. *In this paper we will consider only compactly supported polyhomogeneous conormal functions with integer index sets:* $(z,k) \in E \Rightarrow z \in \mathbb{Z}$. *In particular,* u *may be replaced by* ωu *for a cut-off function* ω *on* $\overline{\mathbb{R}}_+$ (*i.e.,* $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ *such that* $\omega = 1$ *in a neighbourhood of zero*).

In the next part of this section we illustrate the so-called "b-integral" of b-densities. We consider the one-dimensional case, $\overline{\mathbb{R}}_+$; the general case is of an analogous structure.

Let $\mu \in {}^{\mathrm{b}}\Omega^{E}_{\mathrm{pc}}(\overline{\mathbb{R}}_+)$ for an index set E. The integral $\int\limits_{\mathbb{R}_+} \mu$ diverges unless $E \subset \mathbb{N} \times \mathbb{N}_0$. To avoid this problem we "regularize" the latter; we replace usual integrals by b-integrals.

First note that

$$I(x) := \int\limits_x^\infty \mu \in \Omega^E_{\mathrm{pc}}(\overline{\mathbb{R}}_+);$$

(about the calculation see (8) below).

Definition 1.7. *The* b*-integral of* μ*, denoted by*

$$⨍_{\mathbb{R}_+} \mu \tag{5}$$

is defined to be the constant in the asymptotic expansion of $I(x)$ *as* $x \to 0$.

Remark 1.8. (i) *For an integrable* b*-density the* b*-integral coincides with the ordinary integral.*

(ii) *Let us give an explicit expression for* (5). *Using Remark* 1.6 *and the asymptotic expansion of* u *with* $\mu = u(x)\frac{dx}{x}$, *the general terms to be considered are*

$$⨍_{\mathbb{R}_+} \omega(x) x^z \log^k x \, \frac{dx}{x}, \quad z \in \mathbb{Z},\ k \in \mathbb{N}_0. \tag{6}$$

Moreover, let us write

$$u(x) = \omega(x) \cdot \sum_{\substack{(z,k)\in E \\ z \le 0}} a_{zk} x^z \log^k x + r(x), \tag{7}$$

where $r(x) \in \Omega^{\mathbb{N}\times\mathbb{N}_0}_{\mathrm{pc}}(\overline{\mathbb{R}}_+)$; *since* $⨍_{\mathbb{R}_+} r(x)\,\frac{dx}{x} = \int\limits_{\mathbb{R}_+} r(x)\,\frac{dx}{x}$ *we only need to consider* (6) *for* $z \le 0$.

Integration by parts yields

$$\int\limits_x^\infty \omega(y)y^z\log^k y\,\frac{dy}{y} = \begin{cases} \sum\limits_{s=0}^k (-1)^{s+1}\frac{k!}{(k-s)!}z^{-s-1}\left\{x^z\log^{k-s}x + \int\limits_x^\infty \omega'(y)y^z\log^{k-s}y\,dy\right\}, & z<0 \\ -\frac{1}{k+1}\left\{\log^{k+1}x + \int\limits_x^\infty \omega'(y)\log^{k+1}y\,dy\right\}, & z=0 \end{cases}. \quad (8)$$

The integrals on the right side are convergent (*as* $x\to 0$) *and the other terms are polyhomogeneous conormal in* x *without a constant term. Hence*

$$⨍_{\mathbb{R}_+} \omega(x)x^z\log^k x\,\frac{dx}{x} = \begin{cases} \sum\limits_{s=0}^k (-1)^{s+1}\frac{k!}{(k-s)!}z^{-s-1}\int\limits_{\mathbb{R}_+}\omega'(x)x^z\log^{k-s}x\,dx, & z<0 \\ -\frac{1}{k+1}\int\limits_{\mathbb{R}_+}\omega'(x)\log^{k+1}x\,dx, & z=0 \end{cases}. \quad (9)$$

(iii) (*Coordinate invariance*) *In general the* b*-integral depends on the choice of the coordinate* (*i.e., boundary defining function*) x. *Let* $\mu\in {}^{\mathrm{b}}\Omega^E_{\mathrm{pc}}(\overline{\mathbb{R}}_+)$ *and* $\inf E=-m, m\in\mathbb{N}_0$. *Let* x *and* x' *be boundary defining functions to* $\overline{\mathbb{R}}_+ : x' = a(x)x$ *for an* $0<a(x)\in C^\infty(\overline{\mathbb{R}}_+)$. *The* b*-integrals corresponding to* x *and* x' *coincide iff*

$$a(0)=1,\quad a^{(s)}(0)=0, s=1,\dots,m.$$

In fact, by (i) *we have to show the coordinate invariance of the terms in* (6) *for* $z=-m,\dots,0$.

The assertion is an easy consequence from (8). *Let us explain the case* $z=0$. *As shows the second equality of* (8) *by the transition from* x *to* x' *in the asymptotic expansion of* $\int\limits_x^\infty \omega(y)\log^k y\frac{dy}{y}$ *occurs a new constant term,* $-\frac{(\log a(0))^{k+1}}{k+1}$, *which implies the condition* $a(0)=1$.

(iv) *For a meromorphic function* f *we denote by* $\mathrm{R}_0 f(a)$ *the constant in the Laurent expansion of* f *about* a. *Let us show that for* $f\in\Omega^E_{\mathrm{pc}}(\overline{\mathbb{R}}_+)$

$$⨍_{\mathbb{R}_+} f(x)\,\frac{dx}{x} = \mathrm{R}_0(Mf)(0), \quad (10)$$

where Mf *denotes the Mellin transform of* f: $Mf(w)=\int\limits_0^\infty x^w f(x)\,\frac{dx}{x}$. *The assertion is obvious for* r *in* (7), *so we need only consider the terms* $f_{zk}(x) := \omega(x)x^z\log^k x$ *for* $z\le 0$.

Here we consider the case $k=0$. *The Mellin transform of* f_{z0} *is a meromorphic function on* $\mathbb{C}$ *with the single pole at* $-z$ *of order* 1; *hence*

$$\mathrm{R}_0\big(Mf_{z0}\big)(0) = \begin{cases} \lim\limits_{w\to 0}(Mf_{z0})(w), & z<0 \\ \lim\limits_{w\to 0}\big(wMf_{00})(w)\big)', & z=0 \end{cases}.$$

For $\operatorname{Re} w > -z\,(z<0)$ *and* $\operatorname{Re} w > 0$*, integrating by parts, we get*

$$\big(Mf_{z0}\big)(w) = \frac{1}{w+z}\int\limits_0^\infty (x^{w+z})'\omega(x)\,dx = -\frac{1}{w+z}\int\limits_0^\infty x^w\omega'(x)x^z\,dx$$

and

$$w\big(Mf_{00}\big)(w) = \int\limits_0^\infty (x^w)'\omega(x)\,dx = -\int\limits_0^\infty x^w\omega'(x)\,dx$$

respectively. Thus

$$R_0\big(Mf_{z0}\big)(0) = \begin{cases} -\frac{1}{z}\int\limits_0^\infty \omega'(x)x^z\,dx, & z<0 \\ -\int\limits_0^\infty \omega'(x)\log\,x\,dx, & z=0 \end{cases} = \fint_{\mathbb{R}_+} f_{z0}(x)\,\frac{dx}{x},$$

cf. (9).

1.2 Push–Forward Theorem

We now formulate the Melrose's Push–Forward Theorem in a special case of manifolds with corners, in particular, in the case considered in Section 2.

Definition 1.9. (i) *Let* X, Y *be manifolds with corners und* ρ_H *und* ρ'_G *the complete sets of boundary defining functions for the boundary hypersurfaces* $\{H\}$ *of* X *and* $\{G\}$ *of* Y*, respectively. A smooth map* $f : X \to Y$ *is said to be a* b*-map if for any hypersurface* G *of* Y *is*

$$f^*\rho'_G = a_G\prod_H \rho_H^{e(H,G)}, \quad 0 < a_G \in C^\infty(X),$$

with non-negative integers $e(H,G)$*.*

In particular, a b*-map* $f : \overline{\mathbb{R}}_+^2 \to \overline{\mathbb{R}}_+$ (*in local coordinates*) *has the form*

$$f(x,y) = a(x,y)x^\alpha y^\beta \quad \textit{for} \quad 0 < a(x,y) \in C^\infty(\overline{\mathbb{R}}_+^2), \quad \alpha,\beta\in\mathbb{N}_0.$$

(ii) *A* b*-fibration* $f : X \to \overline{\mathbb{R}}_+$ *is a* b*-map such that* $f : f^{-1}(\mathbb{R}_+) \to \mathbb{R}_+$ *is a fibration in the following sense:*
for any $t_0 \in \mathbb{R}_+$ *the preimage* $X_{t_0} := f^{-1}(t_0)$ *is a manifold and a neighbourhood of* X_{t_0} *in* X *can be identified* (*via a diffeomorphism*) *with* $X_{t_0}\times U$ *for some neighbourhood* $U\subset\mathbb{R}_+$ *of* t_0 *such that* $f(x,t) = t$ *for all* $x\in X_t, t\in U$*.*

Let $\mu \in {}^{\mathrm{b}}\Omega^{\mathcal{E}}_{\mathrm{pc}}(X)$ be compactly supported, $f : X \to \overline{\mathbb{R}}_+$ a b-fibration and $\mathcal{E}, f$ satisfy the condition (11) below. The Melrose's Push–Forward Theorem shows that $f_*\mu \in {}^{\mathrm{b}}\Omega^{F}_{\mathrm{pc}}(\overline{\mathbb{R}}_+)$[1] for a resulting Index set F, defined by $\mathcal{E}$ and f, see (12) below.

[1]Let $f : M \to N$ be a smooth map between manifolds. The push–forward $f_*\mu$ of a measure μ on M is the measure on N defined by $(f_*\mu)(V) = \mu(f^{-1}(V))(=$ measure of the union of all fibers over $V), V\subset N$. In terms of integrals, this is equivalent to $\int\limits_N (f_*\mu)\varphi = \int\limits_M \mu f^*\varphi$ for all $\varphi\in C_0^\infty(N)$.

Theorem 1.10. *Let X be a manifold with corners and $f : X \to \overline{\mathbb{R}}_+$ a b-fibration. Let $\mathcal{E} = (E(H))$ (with $E(H)$ an index set to the boundary hypersurface H of X) such that*

$$\inf E(H) > 0 \quad \textit{whenever} \quad e(H,\{0\}) = 0. \tag{11}$$

Then for any compactly supported b-density μ on X, polyhomogeneous with index family $\mathcal{E}$, $f_*\mu$ is a b-density on $\overline{\mathbb{R}}_+$, polyhomogeneous with index family

$$\overline{\bigcup}_{e(H,\{0\})>0}^{H} \big\{ \big(\frac{z}{e(H,\{0\})}, k \big) : (z,k) \in E(H) \big\}; \tag{12}$$

here $E \overline{\cup} F := E \cup F \cup \{(z, k' + k'' + 1) : (z,k') \in E, (z,k'') \in F\}$.

About the conditions of this theorem see [2, Section 3.6].

2 Push–Forwards of b-densities under the map $f : \overline{\mathbb{R}}_+^2 \to \overline{\mathbb{R}}_+, f(x,y) = xy$

Let f be the b-fibration $f : \overline{\mathbb{R}}_+^2 \to \overline{\mathbb{R}}_+, f(x,y) = xy = t$, $\mathcal{E} = (E,F)$ an index family with $E,F \subset \mathbb{Z} \times \mathbb{N}_0$, and μ a compactly supported one in ${}^{\mathrm{b}}\Omega^{\mathcal{E}}_{\mathrm{pc}}(\overline{\mathbb{R}}_+^2)$. According to the Push–Forward Theorem, $f_*\mu \in {}^{\mathrm{b}}\Omega^{G}_{\mathrm{pc}}(\overline{\mathbb{R}}_+)$ with

$$G = E \overline{\cup} F.$$

The purpose of this paper is to compute the asymptotic sum of $f_*\mu$ as $t \to 0$. The consideration will be performed in two steps. First, in Section 2.1, we study smooth b-densities ($E,F \subset \mathbb{N}_0 \times \{0\}$), afterwards, in Section 2.2, we consider a more general case ($E,F \subset \mathbb{Z} \times \mathbb{N}_0$).

2.1 Push–forwards of compactly supported smooth b-densities

Let $\mu = u(x,y)\frac{dx}{x}\frac{dy}{y}$ for $u \in C_0^\infty(\overline{\mathbb{R}}_+^2)$. For a suitable cut-off function ω we can also write $\mu = \omega(x)\omega(y)u(x,y)\frac{dx}{x}\frac{dy}{y}$. Moreover, let $f : \overline{\mathbb{R}}_+^2 \to \overline{\mathbb{R}}_+$ be as above, $f(x,y) = xy$. Then $f_*\mu$ is a smooth b-density with an index set $G \subset \mathbb{N}_0 \times \{0,1\}$:

$$f_*\mu = \int\limits_{f^{-1}(t)} \mu = g(t)\,\frac{dt}{t}$$

with $\Omega^G_{\mathrm{pc}}(\overline{R}_+) \ni g(t) = \int\limits_0^\infty \omega(x)\omega\big(\frac{t}{x}\big)u(x,\frac{t}{x})\,\frac{dx}{x}$. Applying the Mellin transform on g we get

$$F(w) := M_{t\to w}(g(t))(w) = M_{y\to w}\omega(y)\big(M_{x\to w}(\omega(x)u(x,y))(w,y)\big)(w). \tag{13}$$

Let $(j,0)\in E\cap F$. Write the Laurent expansion of

$$h(w,y):=M_{x\to w}(\omega(x)u(x,y))(w,y) \tag{14}$$

close to $-j$:

$$h(w,y)=\frac{h_{-1}(y)}{w+j}+h_0(y)+p(w,y)(w+j)$$

with a family of holomorphic functions $p(w,y), y\in\overline{\mathbb{R}}_+$, in a neighbourhood of $-j$. Let us compute the coefficients $h_{-1}(y)$ and $h_0(y)$. For $\operatorname{Re} w>0$ integration by parts gives

$$h(w,y)=\frac{(-1)^{j+1}}{w(w+1)\cdots(w+j)}\int_0^\infty x^{w+j}\big(\omega(x)u(x,y)\big)^{(j+1)}\,dx,$$

which yields

$$\begin{aligned} h_{-1}(y)=\lim_{w\to -j}(w+j)h(w,y) &= -\frac{1}{j!}\int_0^\infty \big(\omega(x)u(x,y)\big)^{(j+1)}dx \\ &= \frac{u^{(j)}_{x^j}(0,y)}{j!}. \end{aligned}$$

Further,

$$\begin{aligned} h_0(y) &= \mathrm{R}_0\big(M_{x\to w}(\omega(x)u(x,y))(w,y)\big)(-j)=\mathrm{R}_0\big(M_{x\to w}\big(\frac{\omega(x)u(x,y)}{x^j}\big)(w+j,y)\big)(-j) \\ &= \mathrm{R}_0\big(M_{x\to w}\big(\frac{\omega(x)u(x,y)}{x^j}\big)(w,y)\big)(0)=(\text{by Remark 1.8 (iv) })=\fint_{\mathbb{R}_+}\frac{u(x,y)}{x^j}\frac{dx}{x}. \end{aligned}$$

Iterating this process it is not difficult to show that

$$\begin{aligned} F(w)&=M_{y\to w}\big(\omega(y)h(w,y)\big)= \\ &\frac{u^{(2j)}_{x^jy^j}(0,0)}{(j!)^2}\frac{1}{(w+j)^2}+\frac{1}{j!}\left(\fint_{\mathbb{R}_+}\frac{u^{(j)}_{x^j}(0,y)}{y^j}\frac{dy}{y}+\fint_{\mathbb{R}_+}\frac{u^{(j)}_{y^j}(x,0)}{x^j}\frac{dx}{x}\right)\frac{1}{w+j}+q(w) \end{aligned}$$

with a holomorphic function q in a neighborhood of $-j$.

Set

$$\Gamma_\beta:=\{w\in\mathbb{C}:\operatorname{Re} w=\beta\}\quad\text{and}\quad S_\beta:=\{w\in\mathbb{C}:\beta-\frac{1}{2}<\operatorname{Re} w<\beta+\frac{1}{2}\}.$$

We return to the function g as the inverse of the Mellin transform of F:

$$
\begin{aligned}
g(t) &= M^{-1}_{w\to t}\big(F(w)\big)(t) = \frac{1}{2\pi i}\int\limits_{\Gamma_{\frac{1}{2}}} t^{-w}F(w)\,dw \\
&= \frac{1}{2\pi i}\sum_{j=0}^{N-1}\left(\;\int\limits_{\Gamma_{-j+\frac{1}{2}}} t^{-w}F(w)dw - \int\limits_{\Gamma_{-j-\frac{1}{2}}} t^{-w}F(w)dw\right) + \frac{1}{2\pi i}\int_{\Gamma_{-N+\frac{1}{2}}} t^{-w}F(w)dw \\
&= \sum_{j=0}^{N-1}\frac{1}{2\pi i}\int_{\partial S_{-j}} t^{-w}F(w)dw + \frac{1}{2\pi i}\int\limits_{\Gamma_{-N+\frac{1}{2}}} t^{-w}F(w)dw \\
&= \sum_{j=0}^{N-1}\left\{\frac{u^{2j}_{x^jy^j}(0,0)}{(j!)^2}\frac{1}{2\pi i}\int\limits_{\partial S_{-j}}\frac{t^{-w}}{(w+j)^2}dw + \frac{1}{j!}\left(\fint\limits_{\mathbb{R}_+}\frac{u^{(j)}_{y^j}(x,0)}{x^j}\,\frac{dx}{x}\right.\right. \\
&+ \left.\left.\fint\limits_{\mathbb{R}_+}\frac{u^{(j)}_{x^j}(0,y)}{y^j}\,\frac{dy}{y}\right)\frac{1}{2\pi i}\int_{\partial S_{-j}}\frac{t^{-w}}{w+j}dw\right\} + \frac{1}{2\pi i}\int_{\Gamma_{-N+\frac{1}{2}}} t^{-w}F(w)dw \\
&= \sum_{j=0}^{N-1} t^j\left(-\frac{u^{(2j)}_{x^jy^j}(0,0)}{(j!)^2}\log t + \frac{1}{j!}\fint\limits_{\mathbb{R}_+}\frac{u^{(j)}_{y^j}(x,0)}{x^j}\,\frac{dx}{x} + \frac{1}{j!}\fint\limits_{\mathbb{R}_+}\frac{u^{(j)}_{x}(0,y)}{y^j}\,\frac{dy}{y}\right) + t^N\delta(t)
\end{aligned}
$$

with $\delta \in \Omega^{\mathbb{N}_0\times\{0,1\}}_{\text{pc}}(\overline{\mathbb{R}}_+)$; i.e.,

$$
g(t) \sim \sum_{j\ge 0} t^j\left(-\frac{u^{(2j)}_{x^jy^j}(0,0)}{(j!)^2}\log t + \frac{1}{j!}\fint\limits_{\mathbb{R}_+}\frac{u^{(j)}_{y^j}(x,0)}{x^j}\frac{dx}{x} + \frac{1}{j!}\fint\limits_{\mathbb{R}_+}\frac{u^{(j)}_{x}(0,y)}{y^j}\,\frac{dy}{y}\right) \quad \text{as} \quad t\to 0. \tag{15}
$$

2.2 The general case

Now we consider compactly supported b-densities polyhomogeneous with index sets E, F such that $(z,k)\in E$ or $F \Rightarrow z\in\mathbb{Z}, k\in\mathbb{N}_0$. Let $I := \inf E \le \inf F =: J$ and let $(j,k_j)\in E$ and $(j,m_j)\in F$. For $h(w,y)$ as in (14) the Laurent expansion in a neighbourhood of $-j$ is

$$
\begin{aligned}
h(w,y) &= \frac{h_{-k_j-1}(y)}{(w+j)^{k_j+1}} + \frac{h_{-k_j}(y)}{(w+j)^{k_j}} + \ldots + \frac{h_{-1}(y)}{w+j} \\
&+ h_0(y) + h_1(y)(w+j) + \ldots + h_{m_j}(y)(w+j)^{m_j} + (w+j)^{m_j+1}g(w,y)
\end{aligned} \tag{16}
$$

with a family of holomorphic functions $g(w,y), y\in\overline{\mathbb{R}}_+$, in a neighbourhood of $-j$. Write the asymptotic expansion of u at $\mathbb{R}_{+,y}$:

$$
u(x,y) \sim \sum_{j\ge I} x^j\big(\alpha^0_j(y) + \alpha^1_j(y)\,\log x + \cdots + \alpha^{k_j}_j(y)\,\log^{k_j} x\big).
$$

For $k \geq 0$, using some obvious properties of the Mellin transform, we get

$$\begin{aligned} k!\, h_k(y) &= \mathrm{R}_0\big(\big(M_{x\to w}(\omega(x)u(x,y))(w,y)\big)^{(k)}\big)(-j) \\ &= \mathrm{R}_0\big(M_{x\to w}\big(\frac{\omega(x)u(x,y)\log^k x}{x^j}\big)(w+j,y)\big)(-j) \\ &= \mathrm{R}_0\big(M_{x\to w}\big(\frac{\omega(x)u(x,y)\log^k x}{x^j}\big)(w,y)\big)(0) \\ &= \int\!\!\!\!\!\!-_{\mathbb{R}_+} \frac{u(x,y)\log^k x}{x^j}\,\frac{dx}{x}. \end{aligned}$$

For the coefficients of the negative powers of $w+j$, adding the equality $wM\big(u(x)\big)(w) = -M\big(xu'(x)\big)(w)$, we show that

$$h_k(y) = (-1)^{-k-1}(-k-1)!\alpha_j^{-k-1}(y) \quad \text{for} \quad -k_j - 1 \leq k \leq -1. \tag{17}$$

Let us prove (17) for $k = -1$:

$$\begin{aligned} h_{-1}(y) &= \mathrm{R}_0\big((w+j)M\big(\omega(x)u(x,y)\big)(w,y)\big)(-j) \\ &= \mathrm{R}_0\big(w+j)M\big(\frac{\omega(x)u(x,y)}{x^j}\big)(w+j,y)\big)(-j) \\ &= \mathrm{R}_0\big(-M\big(x\big(\frac{\omega(x)u(x,y)}{x^j}\big)'\big)(w,y)\big)(0) \\ &= -\int\!\!\!\!\!\!-_{\mathbb{R}_+} \big(\frac{\omega(x)u(x,y)}{x^j}\big)'\,dx = \alpha_j^0(y). \end{aligned}$$

Putting the values of $h_k(y)$ in (16) we get

$$\begin{aligned} h(w,y) &= \sum_{k=1}^{k_j+1}(-1)^{k-1}(k-1)!\alpha_j^{k-1}(y)\,(w+j)^{-k} \\ &+ \sum_{k=0}^{m_j}\frac{1}{k!}\int\!\!\!\!\!\!-_{\mathbb{R}_+} \frac{u(x,y)\log^k x}{x^j}\frac{dx}{x}\,(w+j)^k \\ &+ (w+j)^{m_j+1}g(w,y). \end{aligned}$$

Write the asymptotic expansion of u at $\mathbb{R}_{+,x}$

$$u(x,y) \sim \sum_{j\geq J} y^j\big(\beta_j^0(x) + \beta_j^1(x)\log y + \cdots + \beta_j^{m_j}(x)\log^{m_j} y\big)$$

and of $\alpha_j^k(y), k = 0, \ldots, k_j$, as $y \to 0$

$$\alpha_j^k(y) \sim \sum_{i\geq J} y^i\big(\gamma_{ji}^{k,0} + \gamma_{ji}^{k,1}\log y + \cdots + \gamma_{ji}^{k,m_i}\log^{m_i} y\big).$$

Let us now consider $M_{y\to w}\big(\omega(y)h(w,y)\big)(w)$. As in the last section, iterating the above discussion, we get

$$\begin{aligned}
M_{y\to w}\big(\omega(y)\alpha_j^{k-1}(y)\big)(w) &= \sum_{m=1}^{m_j+1}(-1)^{m-1}(m-1)!\gamma_{jj}^{k-1,m-1}\,(w+j)^{-m}\\
&+ \sum_{m=0}^{k_j}\frac{1}{m!}\fint_{\mathbb{R}_+}\frac{\alpha_j^{k-1}(y)\log^m y}{y^j}\,\frac{dy}{y}\,(w+j)^m\\
&+ (w+j)^{k_j+1}p_k(w)
\end{aligned}$$

and

$$M_{y\to w}\left(\fint_{\mathbb{R}_+}\frac{\omega(y)u(x,y)\log^k x}{x^j}\,\frac{dx}{x}\right)(w) =$$
$$\sum_{m=1}^{m_j+1}(-1)^{m-1}(m-1)!\fint_{\mathbb{R}_+}\frac{\beta_j^{m-1}(x)\log^k x}{x^j}\,\frac{dx}{x}\,(w+j)^{-m}+q_k(w)$$

with holomorphic functions p_k and q_k in a neighbourhood of $-j$.

Summing up, for F as in (13) we have

$$\begin{aligned}
F(w) &= \sum_{k=1}^{k_j+1}\sum_{m=1}^{m_j+1}(-1)^{k+m}(k-1)!(m-1)!\gamma_{jj}^{k-1,m-1}\,(w+j)^{-(k+m)}\\
&+ \sum_{k=1}^{k_j+1}\sum_{m=0}^{k-1}(-1)^{k-1}\frac{(k-1)!}{m!}\fint_{\mathbb{R}_+}\frac{\alpha_j^{k-1}(y)\log^m y}{y^j}\,\frac{dy}{y}\,(w+j)^{m-k}\\
&+ \sum_{m=1}^{m_j+1}\sum_{k=0}^{m-1}(-1)^{m-1}\frac{(m-1)!}{k!}\fint_{\mathbb{R}_+}\frac{\beta_j^{m-1}(x)\log^k x}{x^j}\frac{dx}{x}\,(w+j)^{k-m}\\
&+ h(w)
\end{aligned}$$

with a holomorphic function h in a neighbourhood of $-j$.

The last step is of analogous structure as in Section 2.1; the asymptotic expansion of g is

$$\begin{aligned}
g(t) \sim\ & \sum_{j\geq J}t^j\left(-\sum_{k=1}^{k_j+1}\sum_{m=1}^{m_j+1}\frac{(k-1)!(m-1)!}{(k+m-1)!}\gamma_{jj}^{k-1,m-1}\log^{k+m-1}t\right)\\
+ & \sum_{j\geq I}t^j\left(\sum_{k=1}^{k_j+1}\sum_{m=0}^{k-1}(-1)^m\frac{(k-1)!}{m!(k-m-1)!}\fint_{R_+}\frac{\alpha_j^{k-1}(y)\log^m y}{y^j}\,\frac{dy}{y}\,\log^{k-m-1}t\right)\\
+ & \sum_{j\geq J}t^j\left(\sum_{m=1}^{m_j+1}\sum_{k=0}^{m-1}(-1)^k\frac{(m-1)!}{k!(m-k-1)!}\fint_{R_+}\frac{\beta_j^{m-1}(x)\log^k x}{x^j}\,\frac{dx}{x}\,\log^{m-k-1}t\right)
\end{aligned}\tag{18}$$

□

2.3 Some conclusions

Let X be a manifold with corner, H a boundary hypersurface and ρ a boundary defining function corresponding to H. Define

$$\begin{aligned}[\rho]_m(X;H) \quad := \quad & \{\rho' : \rho' \quad \text{is a boundary defining function to} \quad H \\ & \text{such that} \quad \rho - \rho' = O(\rho^{m+2}) \quad \text{for} \quad \rho \to 0\}.\end{aligned}$$

Let $\mu = u(x)\frac{dx}{x} \in {}^{\mathrm{b}}\Omega^E_{\mathrm{pc}}(\overline{\mathbb{R}}_{+,x}), E \subset \mathbb{Z} \times \mathbb{N}_0, \inf E = I < 0,$ be compactly supported, $\nu := [x]_{-I}(\overline{\mathbb{R}}_{+,x};\{0\})$ and $x, x' \in \nu$ be two different choice of local coordinates with $x' = a(x)x, 0 < a(x) \in C^\infty(\overline{\mathbb{R}}_{+,x})$. The condition $x, x' \in \nu$ is equivalent to

$$a(0) = 1, a^{(i)}(0) = 0, i = 0, 1, \dots, -I.$$

Thus, using Remark 1.8 (iii), $⨍_{\mathbb{R}_{+,\nu}} \mu$ is well-defined, i.e., is independent of the choice of local coordinate $x \in \nu$. In particular, if $\alpha := [t]_0(\overline{\mathbb{R}}_{+,t};\{0\}), \nu_1 := [x]_{-I}(\overline{\mathbb{R}}^2_+;\overline{\mathbb{R}}_{+,y}), \nu_2 := [y]_{-J}(\overline{\mathbb{R}}^2_+;\overline{\mathbb{R}}_{+,x})$, then the principal part of the asymptotic expansion (18) is independent of particular choices of coordinates $t \in \alpha, x \in \nu_1, y \in \nu_2$.

Remark 2.1. *In the smooth case we have*

$$u(0,0) = \mu|_{(0,0)}, \quad u(0,y)\frac{dy}{y} = \mu|_{\overline{\mathbb{R}}_{+,y}}, u(x,0)\frac{dx}{x} = \mu|_{\overline{\mathbb{R}}_{+,x}}. \tag{19}$$

Let us show the coordinate independence of the restrictions (19). *In fact, if* x, x' *with* $x' = a(x,y)x$ *and* y, y' *with* $y' = b(x,y)y$ *are two boundary defining functions to* $\overline{\mathbb{R}}_{+,y}$ *and* $\overline{\mathbb{R}}_{+,x}$*, respectively, and*

$$\mu = u(x,y)\frac{dx}{x}\frac{dy}{y} = v(x',y)\frac{dx'}{x'}\frac{dy}{y},$$

we have

$$\begin{aligned} u(x,y)\frac{dx}{x}\frac{dy}{y} \quad &= \quad v(a\cdot x,y)\big(\frac{dx}{x} + \frac{a'_x dx}{a}\big)\frac{dy}{y} \\ &= \quad v(a\cdot x,y)\,\frac{dx}{x}\frac{dy}{y} + v(a\cdot x,y)\frac{a'_x}{a}\cdot x\,\frac{dx}{x}\frac{dy}{y},\end{aligned}$$

which gives $u(0,y)\frac{dy}{y} = v(0,y)\frac{dy}{y}$, *i.e., the restriction* $\mu|_{\overline{\mathbb{R}}_{+,y}}$ *is independent of the choice of the boundary defining function to* $\overline{\mathbb{R}}_{+,y}$. *In a similar manner we can argue for* $\mu|_{\overline{\mathbb{R}}_{+,x}}$. *Finally, if*

$$\mu = u(x,y)\,\frac{dx}{x}\frac{dy}{y} = v(x',y')\,\frac{dx'}{x'}\frac{dy'}{y'},$$

the equality

$$\begin{aligned} u(x,y)\frac{dx}{x}\frac{dy}{y} \quad &= \quad v(a\cdot x, b\cdot y)\big(\frac{dx}{x} + \frac{da}{a}\big)\big(\frac{dy}{y} + \frac{db}{b}\big) \\ &= \quad v(a\cdot x, b\cdot y)\big(1 + y\cdot\frac{b'_y}{b} + x\cdot\frac{a'_x}{a} + xy\cdot\frac{a'_x}{a}\frac{b'_y}{b}\big)\frac{dx}{x}\frac{dy}{y}\end{aligned}$$

yields $u(0,0) = v(0,0)$, *the coordinate independence of* $\mu|_{(0,0)}$ *of the choice of boundary defining functions to* $\overline{\mathbb{R}}_{+,x}$ *and* $\overline{\mathbb{R}}_{+,y}$.

We proved the following (coordinate invariant) result:

Theorem 2.2. *Let* $f : \overline{\mathbb{R}}_+^2 \to \overline{\mathbb{R}}_+$ *be a* $(1,1)$*-type* b*-fibration over* $(0,\infty)$. *Let* $\alpha = [t]_0(\overline{\mathbb{R}}_{+,t}, \{0\}), \nu_1 = [x]_0(\overline{\mathbb{R}}_+^2, \overline{\mathbb{R}}_{+,y}), \nu_2 = [y]_0(\overline{\mathbb{R}}_+^2, \overline{\mathbb{R}}_{+,x})$, *satisfying the condition*

$$f^*\alpha = \nu_1 \otimes \nu_2.$$

Let $\mu \in {}^{\mathrm{b}}\Omega^{\mathcal{E}}_{\mathrm{pc}}(\overline{\mathbb{R}}_+^2), \mathcal{E} = (E, F)$ *with* $E, F \in \mathbb{N}_0 \times \{0\}$ *be compactly supported. Then* $f_*\mu \in {}^{\mathrm{b}}\Omega^G_{\mathrm{pc}}(\overline{\mathbb{R}}_+)$ *with*

$$G = E \overline{\cup} F,$$

and the principal term is

$$-\mu|_{(0,0)} \log(\rho_t) + \int_{\mathbb{R}_{+,\rho x}} \mu|_{\mathbb{R}_{+,\rho x}} + \int_{\mathbb{R}_{+,\rho y}} \mu|_{\mathbb{R}_{+,\rho y}}$$

for any $\rho_t \in \alpha, \rho_x \in \nu_1, \rho_y \in \nu_2$ *with* $\rho_t = \rho_x \cdot \rho_y$.

References

[1] J. Brüning and R. Seeley, Regular Singular Asymptotics. *Advances in mathematics* 58, 1985, pp. 133–148.

[2] D. Grieser, Basics of the b-Calculus. *In J.B.Gil et al. (eds.), Approaches to Singular Analysis, Operator Theory: Advances and Applications*, 125. Advances in Partial Differential Equations, Birkhuser, Basel, 2001, pp. 30–84.

[3] D. Grieser and M.J. Gruber, Singular Asymptotics Lemma and Push–Forward Theorem. *In J.B.Gil et al. (eds.), Approaches to Singular Analysis, Operator Theory: Advances and Applications*, 125. Advances in Partial Differential Equations, Birkhuser, Basel, 2001, pp. 117–130.

[4] R.B. Melrose, *Calculus of conormal distributions on manifolds with corners. Int. Math. Res. Not.*, 3, 1992, pp. 51–61.

[5] R.B. Melrose, *Differential analysis on manifolds with corners.* Book in preparation.