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Generalization of an Eneström-Kakeya Type Theorem to the Quaternions

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Abstract. The well-known Eneström-Kakeya theorem states that polynomial $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$, where $0 \leq a_0 \leq a_1 \leq \cdots \leq a_n$, has all of its (complex) zeros in $|z| \leq 1$. Many generalizations of this result exist in the literature. In this paper, we extend one such result to the quaternionic setting and state one of the possible corollaries.

Key Words: Location of Zeros of a Polynomial, Eneström-Kakeya Theorem, Quaternionic Polynomial
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Introduction

The classical Eneström-Kakeya theorem concerns the location of the complex zeros of a real polynomial with nonnegative monotone coefficients. It was independently proved by Gustav Eneström in 1893 [3] and Sōichi Kakeya in 1912 [8].

Theorem 1 Eneström-Kakeya Theorem. *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n (where z is a complex variable) with real coefficients satisfying $0 \leq a_0 \leq a_1 \leq \cdots \leq a_n$, then all the zeros of p lie in $|z| \leq 1$.*

A huge number of generalizations of the Eneström-Kakeya theorem exist. Most of them involve weakening the condition on the coefficients. For a survey of such results up to 2014, see [5]. For example, Gardner and Govil [4], inspired by a result of Aziz and Mohammad [1] for power series, presented the following statement [4, Theorem 8].

Theorem 2 *Let $p(z) = \displaystyle\sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n. If $Re\, a_\nu = \alpha_\nu$ and $Im\, a_\nu = \beta_\nu$ for $\nu = 0, 1, 2, \ldots, n, a_n \neq 0$ and for some ℓ, m and $t \geq 0$,*

$$\alpha_0 \leq t\alpha_1 \leq t^2\alpha_2 \leq \cdots \leq t^\ell \alpha_\ell \geq t^{\ell+1}\alpha_{\ell+1} \geq \cdots \geq t^n \alpha_n,$$

$$\beta_0 \leq t\beta_1 \leq t^2\beta_2 \leq \cdots \leq t^m\beta_m \geq t^{m+1}\beta_{m+1} \geq \cdots \geq t^n\beta_n,$$

then $p(z)$ has all its zeros in $R_1 \leq |z| \leq R_2$, where

$$R_1 = \min\{t|a_0|/(2(t^\ell\alpha_\ell + t^m\beta_m) - (\alpha_0+\beta_0) - t^n(\alpha_n+\beta_n-|a_n|)), t\}$$

and

$$\begin{aligned} R_2 \;=\; & \max\Bigg\{\frac{1}{|a_n|}\Bigg(|a_0|t^{n+1} - t^{n-1}(\alpha_0+\beta_0) - t(\alpha_n+\beta_n) \\ & +(t^2+1)(t^{n-\ell-1}\alpha_\ell + t^{n-m-1}\beta_m) \\ & +(t^2-1)\left(\sum_{j=1}^{\ell-1} t^{n-j-1}\alpha_j + \sum_{j=1}^{m-1} t^{n-m-1}\beta_m\right) \\ & +(1-t^2)\left(\sum_{j=\ell+1}^{n-1} t^{n-j-1}\alpha_j + \sum_{j=m+1}^{n-1} t^{n-j-1}\beta_j\right)\Bigg), \frac{1}{t}\Bigg\} \end{aligned}$$

The quaternions, denoted $\mathbb{H}$ in honor of Rowan William Hamilton who introduced them in 1843, are defined as $\mathbb{H} = \{\alpha+\beta\mathrm{i}+\gamma\mathrm{j}+\delta\mathrm{k} \mid \alpha,\beta,\gamma,\delta \in \mathbb{R}\}$ where $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1$. They are the standard example of a noncommutative division ring. The conjugate of quaternion $q = \alpha+\beta\mathrm{i}+\gamma\mathrm{j}+\delta\mathrm{k}$ is $\overline{q} = \alpha-\beta\mathrm{i}-\gamma\mathrm{j}-\delta\mathrm{k}$ and the modulus of q is $|q| = \sqrt{q\overline{q}} = \sqrt{\alpha^2+\beta^2+\gamma^2+\delta^2}$. Notice that $\mathbb{C} = \{\alpha+\beta\mathrm{i} \mid \alpha,\beta \in \mathbb{R}, \mathrm{i}^2 = -1\}$ is a sub-division ring of $\mathbb{H}$. Since $\mathbb{H}$ lacks commutivity, the factor theorem does not hold. This leads to a somewhat complicated behavior of the zeros of a polynomial of a quaternionic variable. For example, the polynomial q^2+1 has uncountably many zeros, namely, every $q = \beta\mathrm{i}+\gamma\mathrm{j}+\delta\mathrm{k}$ with $\beta^2+\gamma^2+\delta^2 = 1$. Thus by giving results on the location of the quaternionic zeros of a polynomial, we include all (finitely many) complex zeros and potentially infinitely many more quaternionic zeros.

The Eneström-Kakeya theorem has recently been extended to polynomials of a quaternionic variable as follows [2].

Theorem 3 *If $p(q) = \sum_{\nu=0}^{n} q^\nu a_\nu$ is a polynomial of degree n (where q is a quaternionic variable) with real coefficients satisfying $0 \leq a_0 \leq a_1 \leq \cdots \leq a_n$, then all the zeros of p lie in $|q| \leq 1$.*

Since the complex numbers are a subset of the quaternions, this result implies Theorem 1. The purpose of this paper is to extend Theorem 2 to the quaternionic setting.

1 Some Preliminary Results Concerning Functions of a Quaternionic Variable

An analytic theory of functions of a quaternionic variable has recently been developed [6, 7]. In particular, Gentilli and Struppa [7] introduced a maximum modulus theorem for regular functions, a class that includes convergent power series and polynomials (see their Remark 1.3 for a more precise definition of "regular"). They proved the following.

Theorem 4 *Let $B = B(0, r)$ be an open ball in $\mathbb{H}$ with center 0 and radius $r > 0$, and let $f : B \to \mathbb{H}$ be a regular function. If $|f|$ has a maximum at a point $a \in B$, then f is constant on B.*

We can now use Theorem 4 to extend Schwarz's lemma from the complex setting (see [10]) to the quaternionic setting.

Lemma 1 *Let $f(q) = \sum_{\nu=0}^{\infty} q^\nu a_\nu$ for $|q| \leq R$, where the coefficients a_ν, $0 \leq \nu < \infty$, and variable q are quaternions. Suppose $f(0) = a_0 = 0$. Then*

$$|f(q)| \leq \frac{M|q|}{R} \text{ for any } |q| \leq R,$$

where $M = \max_{|q|=R} |f(q)|$.

Proof. Define

$$g(q) = \sum_{\nu=1}^{\infty} q^{\nu-1} a_\nu = \begin{cases} q^{-1} f(q) = \sum_{\nu=1}^{\infty} q^{\nu-1} a_\nu, & \text{for } q \neq 0, \\ a_1, & \text{for } q = 0. \end{cases}$$

Let $R > 0$ and $M = \max_{|q|=R} |f(q)|$. Then

$$\max_{|q|=R} |g(q)| = \max_{|q|=R} \left| \frac{f(q)}{q} \right| = \frac{M}{R}.$$

By Theorem 4 applied to g, $|g(q)| \leq M/R$ for $|q| = r \leq R$. Hence $|g(q)| = |q^{-1} f(q)| \leq M/R$ for $0 < |q| = r \leq R$, or $|f(q)| \leq M|q|/R$ for $0 < |q| = r \leq R$. Since $f(0) = 0$, the result also holds for $q = 0$. □

Let $f(q) = \sum_{i=0}^{n} q^i a_i$ and $g(q) = \sum_{j=0}^{m} q^j b_j$ be two polynomials. The *regular product* of f and g is the polynomial $(f * g)(q) = \sum_{k=0}^{mn} q^k c_k$, where $c_k = \sum_{i=0}^{k} q^i a_i b_{k-i}$ for all k [7]. The absence of commutivity in the quaternions has some unexpected implications (for example, the factor theorem does not hold as mentioned above). In particular, we have the next result concerning the zeros of the regular product of two polynomials [9].

Theorem 5 *Let f and g be quaternionic polynomials. Then $(f * g)(q_0) = 0$ if and only if $f(q_0) = 0$ or ($f(q_0) \neq 0$ implies $g(f(q_0)^{-1} q_0 f(q_0)) = 0$).*

2 Statement and Proof of the Main Result

Our main result is the following:

Theorem 6 *Let* $p(q) = \sum_{\nu=0}^{n} q^\nu a_\nu$ *be a polynomial of degree* n *with quaternionic coefficients* $a_\nu = \alpha_\nu + \beta_\nu \mathrm{i} + \gamma_\nu \mathrm{j} + \delta_\nu \mathrm{k}$, $\nu = 0, 1, 2, \ldots, n$. *If* $a_n \neq 0$ *and for some* ℓ, m, r, s *and* $t \geq 0$, *it holds*

$$\alpha_0 \leq t\alpha_1 \leq t^2\alpha_2 \leq \cdots \leq t^\ell \alpha_\ell \geq t^{\ell+1}\alpha_{\ell+1} \geq \cdots \geq t^n \alpha_n,$$

$$\beta_0 \leq t\beta_1 \leq t^2\beta_2 \leq \cdots \leq t^m \beta_m \geq t^{m+1}\beta_{m+1} \geq \cdots \geq t^n \beta_n,$$

$$\gamma_0 \leq t\gamma_1 \leq t^2\gamma_2 \leq \cdots \leq t^r \gamma_r \geq t^{r+1}\gamma_{r+1} \geq \cdots \geq t^n \gamma_n,$$

$$\delta_0 \leq t\delta_1 \leq t^2\delta_2 \leq \cdots \leq t^s \delta_s \geq t^{s+1}\delta_{s+1} \geq \cdots \geq t^n \delta_n,$$

then $p(q)$ *has all its zeroes in* $R_1 \leq |q| \leq R_2$, *where*

$$\begin{aligned} R_1 &= \min\{t|a_0|/(2(t^\ell\alpha_\ell + t^m\beta_m + t^r\gamma_r + t^s\delta_s) - (\alpha_0 + \beta_0 + \gamma_0 + \delta_0) \\ &\quad -t^n(\alpha_n + \beta_n + \gamma_n + \delta_n - |a_n|)), t\} \end{aligned}$$

and

$$\begin{aligned} R_2 &= \max\left\{\frac{1}{|a_n|}\Bigg(|a_0|t^{n+1} - t^{n-1}(\alpha_0 + \beta_0 + \gamma_0 + \delta_0) - t(\alpha_n + \beta_n + \gamma_n + \delta_n)\right. \\ &\quad +(t^2+1)(t^{n-\ell-1}\alpha_\ell + t^{n-m-1}\beta_m + t^{n-r-1}\gamma_r + t^{n-s-1}\delta_s) \\ &\quad +(t^2-1)\Bigg(\sum_{j=1}^{\ell-1} t^{n-j-1}\alpha_j + \sum_{j=1}^{m-1} t^{n-j-1}\beta_j + \sum_{j=1}^{r-1} t^{n-j-1}\gamma_j \\ &\quad +\sum_{j=1}^{s-1} t^{n-j-1}\delta_j\Bigg) + (1-t^2)\Bigg(\sum_{j=\ell+1}^{n-1} t^{n-j-1}\alpha_j + \sum_{j=m+1}^{n-1} t^{n-j-1}\beta_j \\ &\quad \left.+ \sum_{j=r+1}^{n-1} t^{n-j-1}\gamma_j + \sum_{j=s+1}^{n-1} t^{n-j-1}\delta_j\Bigg)\Bigg), \frac{1}{t}\right\}. \end{aligned}$$

Proof. Define $P(q)$ by the equation

$$\begin{aligned} P(q) &= p(q) * (t-q) \\ &= ta_0 + (ta_1 - a_0)q + (ta_2 - a_1)q^2 + \cdots + (ta_n - a_{n-1})q^n - a_n q^{n+1} \\ &= ta_0 + \sum_{j=1}^{n}(ta_j - a_{j-1})q^j - a_n q^{n+1} \\ &= ta_0 + G_1(q), \end{aligned}$$

where $G_1(q) = \sum_{j=1}^{n}(ta_j - a_{j-1})q^j - a_n q^{n+1}$. By Theorem 5, $p(q)*(t-q) = 0$ if and only if either $p(q) = 0$, or ($p(q) \neq 0$ implies $p(q)^{-1}qp(q) - t = 0$). Notice that $p(q)^{-1}qp(q) - t = 0$ is equivalent to $p(q)^{-1}qp(q) = t$ and if $p(q) \neq 0$, this implies that $q = t$. Thus the only zeros of $p(q) * (t - q)$ are $q = t$ and the zeroes of p. For $|q| = t$, we have

$$\begin{aligned}
|G_1(q)| &\leq \sum_{j=1}^{n} |ta_j - a_{j-1}|t^j + |a_n|t^{n+1} \\
&\leq \sum_{j=1}^{n} |t\alpha_j - \alpha_{j-1}|t^j + \sum_{j=1}^{n} |t\beta_j - \beta_{j-1}|t^j \\
&\quad + \sum_{j=1}^{n} |t\gamma_j - \gamma_{j-1}|t^j + \sum_{j=1}^{n} |t\delta_j - \delta_{j-1}|t^j + |a_n|t^{n+1} \\
&= \sum_{j=1}^{\ell} (t\alpha_j - \alpha_{j-1})t^j + \sum_{j=\ell+1}^{n} (\alpha_{j-1} - t\alpha_j)t^j \\
&\quad + \sum_{j=1}^{m} (t\beta_j - \beta_{j-1})t^j + \sum_{j=m+1}^{n} (\beta_{j-1} - t\beta_j)t^j \\
&\quad + \sum_{j=1}^{r} (t\gamma_j - \gamma_{j-1})t^j + \sum_{j=r+1}^{n} (\gamma_{j-1} - t\gamma_j)t^j \\
&\quad + \sum_{j=1}^{s} (t\delta_j - \delta_{j-1})t^j + \sum_{j=s+1}^{n} (\delta_{j-1} - t\delta_j)t^j + |a_n|t^{n+1} \\
&= -t(\alpha_0 + \beta_0 + \gamma_0 + \delta_0) + 2(t^{\ell+1}\alpha_\ell + t^{m+1}\beta_m + t^{r+1}\gamma_r + t^{s+1}\delta_s) \\
&\quad -t^{n+1}(\alpha_n + \beta_n + \gamma_n + \delta_n - |a_n|) = M_1.
\end{aligned}$$

Applying Lemma 1 to $G_1(q)$, we get

$$|G_1| \leq \frac{M_1|q|}{t} \quad \text{for } |q| \leq t,$$

which implies

$$\begin{aligned}
|P(q)| &= |-ta_0 + G_1(q)| \\
&\geq t|a_0| - |G_1(q)| \\
&\geq t|a_0| - \frac{M_1|q|}{t} \quad \text{for } |q| \leq t.
\end{aligned}$$

Therefore, if $|q| < R_1 = \min\{(t^2|a_0|/M_1), t\}$, then $P(q) \neq 0$ and hence $p(q) \neq 0$. Next we want to show that $p(q) \neq 0$ if $|q| > R_2$. For this, we consider once again

$$\begin{aligned}
P(q) &= p(q) * (t - q) = ta_0 + \sum_{j=1}^{n} (ta_j - a_{j-1})q^j - a_n q^{n+1} \\
&= -a_n q^{n+1} + G_2(q),
\end{aligned}$$

where $G_2(q) = ta_0 + \sum_{j=1}^{n}(ta_j - a_{j-1})q^j$. Then

$$\left|q^n G_2\left(\frac{1}{q}\right)\right| = \left|ta_0 q^n + \sum_{j=1}^{n}(ta_j - a_{j-1})q^{n-j}\right|,$$

and for $|q| = t$, we have

$$\begin{aligned}
\left|q^n G_2\left(\frac{1}{q}\right)\right| &\leq |a_0|t^{n+1} + \sum_{j=1}^{n}|ta_j - a_{j-1}|t^{n-j} \\
&\leq |a_0|t^{n+1} + \sum_{j=1}^{n}|t\alpha_j - \alpha_{j-1}|t^{n-j} + \sum_{j=1}^{n}|t\beta_j - \beta_{j-1}|t^{n-j} \\
&\quad + \sum_{j=1}^{n}|t\gamma_j - \gamma_{j-1}|t^{n-j} + \sum_{j=1}^{n}|t\delta_j - \delta_{j-1}|t^{n-j} \\
&= |a_0|t^{n+1} - t^{n-1}(\alpha_0 + \beta_0 + \gamma_0 + \delta_0) - t(\alpha_n + \beta_n + \gamma_n + \delta_n) \\
&\quad +(t^2+1)(t^{n-\ell-1}\alpha_\ell + t^{n-m-1}\beta_m + t^{n-r-1}\gamma_r + t^{n-s-1}\delta_s) \\
&\quad +(t^2-1)\Bigg(\sum_{j=1}^{\ell-1} t^{n-j-1}\alpha_j + \sum_{j=1}^{m-1} t^{n-j-1}\beta_j \\
&\quad + \sum_{j=1}^{r-1} t^{n-j-1}\gamma_j + \sum_{j=1}^{s-1} t^{n-j-1}\delta_j\Bigg) \\
&\quad +(1-t^2)\Bigg(\sum_{j=\ell+1}^{n-1} t^{n-j-1}\alpha_j + \sum_{j=m+1}^{n-1} t^{n-j-1}\beta_j \\
&\quad + \sum_{j=r+1}^{n-1} t^{n-j-1}\gamma_j + \sum_{j=s+1}^{n-1} t^{n-j-1}\delta_j\Bigg) \\
&= M_2.
\end{aligned}$$

By Theorem 4, it follows that

$$\left|q^n G_2\left(\frac{1}{q}\right)\right| \leq M_2 \quad \text{for } |q| \leq t,$$

which implies, by replacing q with $1/q$, that

$$|G_2(q)| \leq M_2|q|^n \quad \text{for } |q| \geq \frac{1}{t}.$$

Hence,

$$|P(q)| = |-a_n q^{n+1} + G_2(q)| \geq |a_n||q|^{n+1} - M_2|q|^n$$

for $|q| \geq 1/t$. Thus $|P(q)| \geq |q|^n(|a_n||q| - M_2)$. Therefore, if $|q| > R_2 = \max\{M_2/|a_n|, 1/t\}$, then $P(q) \neq 0$ and therefore $p(q) \neq 0$. The theorem is proved. $\square$

Since the quaternionic zeros include all of the complex zeros, Theorem 6 generalizes Theorem 2. By adjusting the values of ℓ, m, r, s, and t we can extract a number of corollaries. For example, with $\ell = m = r = s = n$ and $t = 1$ (that is, imposing a condition of monotonicity on the parts of the coefficients) we get the next corollary.

Corollary 1 *Let* $p(z) = \sum_{\nu=0}^{n} a_\nu q^\nu, a_n \neq 0$ *and* $a_\nu = \alpha_\nu + \beta_\nu \mathrm{i} + \gamma_\nu \mathrm{j} + \delta_\nu \mathrm{k}$, $\nu = 0, 1, 2, \ldots, n$. *If*

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n,\ \beta_0 \leq \beta_1 \leq \cdots \leq \beta_n,$$

$$\gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_n,\ \text{and } \delta_0 \leq \delta_1 \leq \cdots \leq \delta_n,$$

then $p(z)$ *has all its zeros in*

$$\frac{|a_0|}{|a_n| - (\alpha_0 + \beta_0 + \gamma_0 + \delta_0) + (\alpha_n + \beta_n + \gamma_n + \delta_n)} \leq |z|$$
$$\leq \frac{|a_n| - (\alpha_0 + \beta_0 + \gamma_0 + \delta_0) + (\alpha_n + \beta_n + \gamma_n + \delta_n)}{|a_n|}.$$

Corollary 1 is related to Theorem 9 of [2]; the outer radius of these two results are the same but Corollary 1 also gives an inner radius of the zero containing region.

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