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# Boolean Centre of a C-algebra

S. Kalesha Vali*, P. Sundarayya** and U.M. Swamy***

* *Department of Basic Sciences and Humanities, GITAM Institute of Technology, GITAM University, Visakhapatnam, Andhra Pradesh, India.*
*vali312@gitam.edu*

** *Department of Basic Sciences and Humanities, GITAM Institute of Technology, GITAM University, Visakhapatnam, Andhra Pradesh, India.*
*psundarayya@yahoo.co.in*

*** *Gayatri Vidya Parishad College of Engineering, Visakhapatnam, Andhra Pradesh, India.*
*umswamy@yahoo.com*

**Abstract**

There is a precise characterisation of factor congruences on a C-algebra with meet identity $T$. The characterisation of such congruences on a C-algebra with out $T$ is a difficult task. In this paper, we make such an attempt and we characterise the factor congruences on a C-algebra $A$ and identify these with certain elements or sets of elements of $A$.

*Key Words:* C-algebra, Centre, Factor congruence, Balanced Congruence, Boolean centre.
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# Introduction

In [2] Guzman and Squier introduced the variety of C-algebras as the variety generated by the three element algebra $C = \{T, F, U\}$, which is the algebraic form of the three valued conditional logic. They proved that $C$ and the two element Boolean algebra $B = \{T, F\}$ are the only subdirectly irreducible C-algebras and that the variety of C-algebras is a minimal cover of the variety of Boolean algebras. Later in [3] G.C.Rao and P.Sundarayya defined different partial orders on a C-algebra and studied their properties and gave a number of equivalent conditions in terms of this partial ordering for a C-algebra to become a Boolean Algebra. In [6], Swamy and Murthy have proved that the set of all balanced factor congruences whose direct complements are also balanced, forms a Boolean permutable sublattice of the lattice Con$(A)$ of congruences on $A$, called Boolean centre and is denoted by $\mathcal{B}(A)$. In [7] U.M.Swamy et.al.,introduced the concept of the Centre, denoted by $\mathbb{B}(A)$ of a C-algebra $A$ with $T$. If $A$ is a C-algebra with $T$, then they proved that every factor congruence on $A$ is of the form $\theta_a$ for some $a \in \mathbb{B}(A)$ also, proved that $a \mapsto \theta_a$ is an isomorphism of $\mathbb{B}(A)$ on to $\mathcal{B}(A)$ of [6] and thus the precise characterization of factor congruences on a C-algebra with $T$. The characterisation of such congruences on a C-algebra with out $T$ is a difficult task. In this paper, we make such an attempt and we characterise the factor congruences on a C-algebra $A$ and identify these with certain elements or sets of elements of $A$. Finally we proved that the Boolean algebras $\mathfrak{B} = \{s \in \prod_{a \in A} \mathbb{B}(A_a) \mid \alpha_{a,b}(s_b) = s_a$, whenever $a \leq_* b\}$, $\mathcal{B}(A)$ and $\mathbb{B}(A)$ are all isomorphic to each other.

# 1 C-algebra

In this section we recall the definition of a C-algebra and some results from [2],[3],[4] and [7]. Let us start with the definition of a C-algebra.

**Definition 1.1:**[2] By a C-algebra we mean an algebra of type $(2, 2, 1)$ with binary operations $\wedge$ and $\vee$ and unary operation $'$ satisfying the following identities.

(1) $x'' = x$ (2) $(x \wedge y)' = x' \vee y'$

(3) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ (4) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

(5) $(x \vee y) \wedge z = (x \wedge z) \vee (x' \wedge y \wedge z)$ (6) $x \vee (x \wedge y) = x$

(7) $(x \wedge y) \vee (y \wedge x) = (y \wedge x) \vee (x \wedge y)$.

**Example 1.2:**[2] The three element algebra $C = \{T, F, U\}$ with the operations given by the following tables is a C-algebra

| $\wedge$ | T | F | U |
|---|---|---|---|
| T | T | F | U |
| F | F | F | F |
| U | U | U | U |

| $\vee$ | T | F | U |
|---|---|---|---|
| T | T | T | T |
| F | T | F | U |
| U | U | U | U |

| $x$ | $x'$ |
|---|---|
| T | F |
| F | T |
| U | U |

**Note 1.3:**[2] The identities 1.1(1), 1.1(2) imply that the variety of C-algebras satisfies all the dual statements of 1.1(3) to 1.1(7). $\wedge$ and $\vee$ are not commutative in $C$. The ordinary distributive law of $\wedge$ over $\vee$ fails in $C$. Every Boolean algebra is a C-algebra.

Note that $C$ always denote the three element C-algebra $\{T, F, U\}$ and $B$ always denote the two element Boolean algebra $\{T, F\}$. $B$ is the only C-algebra of order two. There can be at most one element $x$ satisfying $x' = x$. This element, if it exists, is denoted by U.If a C-algebra $(A, \wedge, \vee, ')$ has an identity for $\wedge$, then it is unique and is denoted by $T$. In this case we say that $A$ is a C-algebra with $T$. We denote $T'$ by $F$.

Now we give some results on C-algebra collected from [2],[3],[4] and [7].

**Lemma 1.4:** Every C-algebra satisfies the following identities:

(1) $x \wedge x = x$ (2) $x \wedge x' = x' \wedge x$

(3) $x \wedge y \wedge x = x \wedge y$ (4) $x \wedge x' \wedge y = x \wedge x'$

(5) $x \wedge y = (x' \vee y) \wedge x$ (6) $x \wedge y = x \wedge (y \vee x')$

(7) $x \wedge y = x \wedge (x' \vee y)$ (8) $x \wedge y \wedge x' = x \wedge y \wedge y'$

(9) $(x \vee y) \wedge x = x \vee (y \wedge x)$ (10) $x \wedge (x' \vee x) = (x' \vee x) \wedge x = (x \vee x') \wedge x = x$.

We recollect the fundamental congruence corresponding to an element in a C-algebra, defined in [2].

**Definition 1.5:**[2] For any element $x$ of a C-algebra $A$, $\theta_x = \{(a, b) \in A \times A \mid x \wedge a = x \wedge b\}$ is a congruence on $A$.

**Lemma 1.6:** Let $A$ be a C-algebra and $x, y \in A$. Then the following hold.

(1) $(x \wedge y, y) \in \theta_x$ (2) $(y \wedge x, y) \in \theta_x$

(3) $(x \wedge y, y \wedge x) \in \theta_x$. (4) $\theta_x \cap \theta_y \subseteq \theta_{x \vee y} \subseteq \theta_x$

(5) $\theta_{x \wedge y} = \theta_{y \wedge x}$ (6) $\theta_{x \wedge y} = \theta_x \vee \theta_y = \theta_y \circ \theta_x \circ \theta_y = \theta_x \circ \theta_y \circ \theta_x$.

# 2 Factor Congruences

In this section section we shall discuss various properties of factor congruences on a C-algebra and identify certain elements or set of elements of the C-algebra with the factor congruences. First we recall the following.

**Definition 2.1:** A congruence $\theta$ on a C-algebra is called a factor congruence if there exist a congruence $\phi$ on $A$ such that $\theta \cap \phi = \Delta_A$ and $\theta \circ \phi = A \times A$; in this case $\phi$ is called a direct complement of $\theta$.

In [7], they specialized factor congruences on a C-algebra with $T$, where $T$ is the identity for the operation $\wedge$ in $A$. We begin with the following which are taken from [7].

**Theorem 2.2:**[7] Let $A$ be a C-algebra with $T$ and define $\mathbb{B}(A) = \{a \in A \mid a \vee a' = T\}$. Then $\mathbb{B}(A)$ is a Boolean algebra under the operations induced by those on $A$, in which $T$ and $F$ are largest and least elements respectively.

**Definition 2.3:**[7] For any C-algebra $A$ with $T$, $\mathbb{B}(A)$ is called the centre of $A$.

**Theorem 2.4:**[7] Let $A$ be a C-algebra with $T$, and $\theta$ is a congruence on $A$. Then $\theta$ is a factor congruence on $A$ if and only if $\theta = \theta_a$ for some $a \in \mathbb{B}(A)$.

**Theorem 2.5:**[7] Let $A$ be a C-algebra with $T$. For any $a, b \in \mathbb{B}(A)$, the following hold.

(1) $\theta_a \cap \theta_b = \theta_{a \vee b}$ (2) $\theta_a \circ \theta_b = \theta_{a \wedge b} = \theta_a \vee \theta_b$

(3) $\theta_T = \Delta_A$ (4) $\theta_F = A \times A$.

A congruence $\theta$ on an (universal) algebra $A$ is called balanced if $(\theta \vee \phi) \cap (\theta \vee \phi') = \theta$ for any direct factor congruences $\phi$ and any of its direct complements $\phi'$ on $A$. In [7] it is proved that if $A$ is a C-algebra with $T$ then the set of all factor congruences on $A$ is a Boolean algebra and is isomorphic with Boolean algebra $\mathbb{B}(A)$. Further, $\theta \circ \phi = \phi \circ \theta$ for all factor congruences $\theta$ and $\phi$ on $A$ also proved that every factor congruence on $A$ is balanced.

Next let us recall the following from [4].

**Theorem 2.6:**[4] Let $A$ be a C-algebra and $a \in A$. Let $A_a = \{x \in A \mid a \wedge x = x\} = \{a \wedge y \mid y \in A\}$. Then $A_a$ is closed under the operations $\wedge$ and $\vee$. Also, for any $x \in A_a$ define $x^a = a \wedge x'$. Then $(A_a, \wedge, \vee, {}^a)$ is a C-algebra with $T$ (here, $a$ itself is the identity for $\wedge$ in $A_a$; that is $T$ in $A_a$).

**Lemma 2.7:** Let $\theta$ be a congruence on A. Then $\theta \cap (A_a \times A_a)$ is a congruence on $A_a$, for each $a \in A$.

**Proof:** Fix $a \in A$. Since $\theta$ is an equivalence relation on A, $\theta \cap (A_a \times A_a)$is an equivalence relation on $A_a$. Let $(x, y), (z, t) \in \theta \cap (A_a \times A_a)$.
Since $x, y, z, t \in A_a, x \wedge z, y \wedge t \in A_a$ and hence $(x \wedge z, y \wedge t) \in \theta \cap (A_a \times A_a)$

$$\begin{aligned}
\text{Now}(x, y) \in \theta \;&\Rightarrow\; (x', y') \in \theta \\
&\Rightarrow\; (a \wedge x', a \wedge y') \in \theta \text{ and } (a \wedge x', a \wedge y') \in A_a \times A_a \\
&\Rightarrow\; (a \wedge x', a \wedge y') \in \theta \cap (A_a \times A_a) \\
&\Rightarrow\; (x^a, y^a) \in \theta \cap (A_a \times A_a) \;\; (\text{since } x^a = a \wedge x' \text{ in } A_a)
\end{aligned}$$

Therefore, $\theta \cap (A_a \times A_a)$ is compatible with the binary operation $\wedge$ and the unary operation ${}^a$ on $A_a$. By the De Morgan laws and the property that $(x^a)^a = x$ for all $x \in A_a$, it follows that $\theta \cap (A_a \times A_a)$ is compatible with $\vee$ also. Thus $\theta \cap (A_a \times A_a)$ is a congruence on $A_a$.

**Lemma 2.8:** Let $\theta$ be a factor congruence on a C-algebra $A$. Then $\theta \cap (A_a \times A_a)$ is a factor congruences on $A_a$.

**Proof:** Since $\theta$ is a factor congruence on $A$, there is a congruence $\theta'$ on $A$ such that $\theta \cap \theta' = \Delta_A$ and $\theta \circ \theta' = A \times A (= \theta' \circ \theta)$.

$$\begin{aligned}
\text{Consider, } [\theta \cap (A_a \times A_a)] \cap [\theta' \cap (A_a \times A_a)] &= (\theta \cap \theta') \cap (A_a \times A_a) \\
&= \Delta \cap (A_a \times A_a) \\
&= \Delta_{A_a}, \text{ the diagonal on } A_a.
\end{aligned}$$

Observe that every element in $A_a$ is in the form $a \wedge x$ for some $x \in A$
Now, let $(a \wedge x, a \wedge y) \in A_a \times A_a$. Then $(a \wedge x, a \wedge y) \in A \times A = \theta' \circ \theta$ which implies that there exists a $z \in A$ such that $(a \wedge x, z) \in \theta$ and $(z, a \wedge y) \in \theta'$.
Now, $(a \wedge x, a \wedge z) \in \theta$ and $(a \wedge z, a \wedge y) \in \theta'$ and $a \wedge z \in A_a$. and hence $(a \wedge x, a \wedge y) \in$

$[\theta' \cap (A_a \times A_a)] \circ [\theta \cap (A_a \times A_a)]$. Therefore $[\theta \cap (A_a \times A_a)] \circ [\theta' \cap (A_a \times A_a)] = A_a \times A_a$. Thus $\theta \cap (A_a \times A_a)$ is a factor congruence on $A_a$ and $\theta' \cap (A_a \times A_a)$ is a direct complement of $\theta \cap (A_a \times A_a)$.
Since $A_a$ is a C-algebra with $T$ every factor congruence is balanced [7]. Hence we have the following.

**Theorem 2.9:** If $\theta$ is a factor congruence on $A$, then $\theta \cap (A_a \times A_a)$ is a balanced factor congruence on $A_a$ for each $a \in A$ and there exists unique $s_a \in \mathbb{B}(A_a)$ such that $\theta \cap (A_a \times A_a) = \theta_{s_a} := \{(x, y) \in A_a \times A_a \mid s_a \wedge x = s_a \wedge y\}$.

Let us recall from [8] that the operation $*$ defined on a C-algebra $A$ by
$a * b = (a \wedge b) \vee (b \wedge a)$ is associative, commutative and idempotent on $A$ thus $(A, *)$ is a semilattice. $\leq_*$ is an induced partial order of the semilattice $(A, *)$ (that is $x \leq_* y$ if and only if $x * y = x$.

**Lemma 2.10:** Let $a$ and $b$ be elements in a C-algebra $A$ such that $a \leq_* b$. Then the following hold.
(1) $a \wedge b = a$
(2) The map $\alpha_{a,b} : A_b \to A_a$ defined by $\alpha_{a,b}(x) = a \wedge x$ for all $x \in A_b$, is a homomorphism of C-algebras.
(3) $\alpha_{a,b}(\mathbb{B}(A_b)) \subseteq \mathbb{B}(A_a)$
(4) If $a \leq_* b \leq_* c$ then $\alpha_{a,b} \circ \alpha_{b,c} = \alpha_{a,c}$
(5) $\alpha_{a,a}$ is the identity map on $A_a$.

**Proof:** We have $a \leq_* b$; that is, $a = a * b = (a \wedge b) \vee (b \wedge a)$. Now,

$$\begin{aligned} a \wedge b &= (a * b) \wedge b \\ &= [(a \wedge b) \vee (b \wedge a)] \wedge b \\ &= (a \wedge b \wedge b) \vee [(a \wedge b)' \wedge (b \wedge a) \wedge b] \\ &= (a \wedge b) \vee [(a \wedge b)' \wedge (b \wedge a)] \\ &= (a \wedge b) \vee (b \wedge a) \\ &= a * b = a. \end{aligned}$$

(2) Let $x, y \in A_b$. Then
$\alpha_{a,b}(x \wedge y) = a \wedge (x \wedge y) = (a \wedge x) \wedge (a \wedge y) = \alpha_{a,b}(x) \wedge \alpha_{a,b}(y)$.
and $\alpha_{a,b}(x \vee y) = a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y) = \alpha_{a,b}(x) \vee \alpha_{a,b}(y)$.

$$\begin{aligned} \text{Also, } \alpha_{a,b}(x^b) &= a \wedge x^b \\ &= a \wedge b \wedge x' \\ &= a \wedge x' && \text{(by (1), } a \wedge b = a) \\ &= a \wedge (a' \vee x') && \text{(by lemma 1.4(7))} \\ &= a \wedge (a \wedge x)' \\ &= (a \wedge x)^a \\ &= (\alpha_{a,b}(x))^a \end{aligned}$$

Therefore $\alpha_{a,b}$ is a homomorphism of C-algebras.
(3) Let $x \in \mathbb{B}(A_b)$. Then $x \vee x^b = b$ and therefore $b = x \vee (b \wedge x')$
Now, $b = b \wedge b = b \wedge (x \vee (b \vee x')) = (b \wedge x) \vee (b \wedge x') = b \wedge (x \vee x') \quad \rightarrow (1)$.

$$\begin{aligned} \text{Now,}\alpha_{a,b}(x) \vee (\alpha_{a,b}(x))^a &= (a \wedge x) \vee (a \wedge x)^a \\ &= (a \wedge x) \vee (a \wedge x') \\ &= a \wedge (x \vee x') \\ &= (a \wedge b) \wedge (x \vee x') \\ &= a \wedge [b \wedge (x \vee x')] \\ &= a \wedge b \quad \text{(since by (1))} \\ &= a, \text{ which is the } T \text{ in } A_a. \end{aligned}$$

Therefore $\alpha_{a,b}(x) \in \mathbb{B}(A_a)$. Thus $\alpha_{a,b}(\mathbb{B}(A_b)) \subseteq \mathbb{B}(A_a)$.
(4) $[\alpha_{a,b} \circ \alpha_{b,c}](x) = \alpha_{a,b}(\alpha_{b,c}(x)) = \alpha_{a,b}(b \wedge x)) = a \wedge b \wedge x = a \wedge x = \alpha_{a,c}(x)$.
Therefore $a \leq_* b \leq_* c \Rightarrow \alpha_{a,b} \circ \alpha_{b,c} = \alpha_{a,c}$.
(5) $\alpha_{a,a}(x) = a \wedge x = x$ for all $x \in A_a$.

**Theorem 2.11:** Let $\theta$ be a factor congruence on a C-algebra $A$ and $a, b \in A$ such that $a \leq_* b$. Let $\theta \cap (A_a \times A_a) = \theta_{s_a},\ s_a \in \mathbb{B}(A_a)$ and $\theta \cap (A_b \times A_b) = \theta_{s_b},\ s_b \in \mathbb{B}(A_b)$. Then the homomorphism $\alpha_{a,b} : A_b \rightarrow A_a$ carries $s_b$ to $s_a$; that is, $a \wedge s_b = s_a$.

**Proof:** Since $\alpha_{a,b}(\mathbb{B}(A_b)) \subseteq \mathbb{B}(A_a)$, it follows that $a \wedge s_b \in \mathbb{B}(A_a)$. By the uniqueness of $s_a$ (theorem 2.9), it is enough if we prove the equality $\theta_{a \wedge s_b} = \theta_{s_a}$ on $A_a$. First, we have that $(b, s_b) \in \theta_{s_b}$ (since $b$ is the identity for $\wedge$ on $A_b$) and hence $(b, s_b) \in \theta_{s_b} = \theta \cap (A_b \times A_b) \subseteq \theta$ and therefore $(b, s_b) \in \theta$. This implies that $(b \wedge x, s_b \wedge x) \in \theta$ for all $x \in A$. Now, for any $x \in A_a$, we have $(x, a \wedge s_b \wedge x) = (a \wedge b \wedge x, a \wedge s_b \wedge x) \in \theta. \quad \rightarrow (1)$
Therefore, if $(x, y) \in \theta_{a \wedge s_b}$, then $a \wedge s_b \wedge x = a \wedge s_b \wedge y$ and hence $(x, y) \in \theta$ (from (1)). Thus $\theta_{a \wedge s_b} \subseteq \theta \cap (A_a \times A_a) = \theta_{s_a}$. On the other hand,

$$\begin{aligned} (x, y) \in \theta_{s_a} &\Rightarrow (x, y) \in \theta \cap (A_a \times A_a) \\ &\Rightarrow (b \wedge x, b \wedge y) \in \theta \cap (A_b \times A_b) = \theta_{s_b} \\ &\Rightarrow s_b \wedge b \wedge x = s_b \wedge b \wedge y \\ &\Rightarrow s_b \wedge x = s_b \wedge y \\ &\Rightarrow a \wedge s_b \wedge x = a \wedge s_b \wedge y \\ &\Rightarrow (x, y) \in \theta_{a \wedge s_b} \end{aligned}$$

Therefore $\theta_{s_a} \subseteq \theta_{a \wedge s_b}$. Thus $\theta_{s_a} = \theta_{a \wedge s_b}$ and hence $s_a = a \wedge s_b$ that is, $s_a = \alpha_{a,b}(s_b)$.

For each element $a$ in a C-algebra, we know that $A_a$ is a C-algebra with $T$ and $\mathbb{B}(A_a)$ is a Boolean algebra under the operations induced by those in $A_a$, where $\mathbb{B}(A_a) = \{x \in A_a \mid x \vee x^a = a\}$. Therefore the direct product $\prod_{a \in A} \mathbb{B}(A_a)$ is also a Boolean algebra under the pointwise operations. In the following, we identify a subalgebra of this product.

**Theorem 2.12:** Let $A$ be a C-algebra and
$\mathfrak{B} = \{s \in \prod_{a \in A} \mathbb{B}(A_a) \mid \alpha_{a,b}(s_b) = s_a, \text{ whenever } a \leq_* b\}$. Then $\mathfrak{B}$ is a Boolean algebra under

the pointwise operations.

**Proof:** We have to simply prove that $\mathfrak{B}$ is a subalgebra of the product $\prod_{a\in A} \mathbb{B}(A_a)$ of Boolean algebras. Recall that $a$ is the largest element (identity for $\wedge$) in $\mathbb{B}(A_a)$ and hence the identity map $i$, defined by $i_a = a$ for any $a \in A$, is the largest element in the product $\prod_{a\in A} \mathbb{B}(A_a)$. Also, $i \in \mathfrak{B}$; for, if $a \leq_* b$ in $A$, then $a \wedge b = a$ and hence $\alpha_{a,b}(i_b) = \alpha_{a,b}(b) = a \wedge b = a$. Further the complement $a^a$ of $a$ in $A_a$ is $a^a = a \wedge a'$. Therefore $a \wedge a'$ is the smallest element in $\mathbb{B}(A_a)$. If $0 \in \prod_{a\in A} \mathbb{B}(A_a)$ is defined by $0_a = a \wedge a'$, for all $a \in A$, then 0 is the smallest element in $\prod_{a\in A} \mathbb{B}(A_a)$. Also, whenever $a \leq_* b$, $\alpha_{a,b}(0_b) = \alpha_{a,b}(b \wedge b') = a \wedge b \wedge b' = a \wedge b \wedge a'$( by lemma 1.4(8))$= a \wedge a' = 0_a$.
Therefore $0 \in \mathfrak{B}$. Now, since $\alpha_{a,b} : A_b \to A_a$ is a homomorphism of
C-algebras, its restriction to $\mathbb{B}(A_b)$ is a homomorphism of (Boolean algebras) $\mathbb{B}(A_b)$ into $\mathbb{B}(A_a)$. From this it follows that $\mathfrak{B}$ is a subalgebra of $\prod_{a\in A} \mathbb{B}(A_a)$. Thus $\mathfrak{B}$ is a Boolean algebra under the pointwise operations.

It is known from [6], that $\mathcal{B}(A)$ is a Boolean algebra under the usual operations on the lattice Con$(A)$ of congruences on $A$. Infact, $\mathcal{B}(A)$ is a bounded distributive and permutable sublattice of Con$(A)$ and is closed under complements. Now we prove the following.

**Theorem 2.13:** Let $A$ be a C-algebra and $\mathcal{B}(A)$ be the Boolean algebra of all balanced factor congruences which admit balanced direct complements. Let $\mathfrak{B}$ be the Boolean algebra described in theorem 2.12. Then $\mathcal{B}(A)$ can be embedded in the Boolean algebra $\mathfrak{B}$.

**Proof:** Let $\theta$ be a factor congruence on $A$. Then, by theorem 2.9, for each $a \in A$, there exists unique $s_a \in \mathbb{B}(A_a)$ such that
$\theta \cap (A_a \times A_a) = \theta_{s_a} = \{(x, y) \in A_a \times A_a \mid s_a \wedge x = s_a \wedge y\}$. Now, define $f : \mathcal{B}(A) \to \mathfrak{B}$ by $f(\theta) = s'$, where $s \in \prod_{a\in A} \mathbb{B}(A_a)$ is given by the relation $\theta \cap (A_a \times A_a) = \theta_{s_a}$ for each $a \in A$. We shall verify that $f$ is an embedding of Boolean algebras. Recall that $a \wedge a'$ and $a$ are respectively the least and greatest elements in $\mathbb{B}(A_a)$, for each $a \in A$. Also, $\Delta_A$ and $A \times A$ are respectively the least and greatest elements in $\mathcal{B}(A)$.
Further, for any $a \in A$, $\theta_{a\wedge a'} = A_a \times A_a$ (since $a \wedge a' \wedge x = a \wedge a'$, for all $x \in A_a$) and $\theta_a = \Delta_{A_a}$ (since $a \wedge x = x$, for all $x \in A_a$). All these imply that the least (greatest) element of $\mathcal{B}(A)$ carried to that of $\mathfrak{B}$. Next, let $\theta, \phi \in \mathbb{B}(A_a)$ and $f(\theta) = s'$ and $f(\phi) = t'$. Then $\theta \cap (A_a \times A_a) = \theta_{s_a}$ and $\theta \cap (A_a \times A_a) = \theta_{t_a}$.
Now, $(\theta \cap \phi) \cap (A_a \times A_a) = \theta_{s_a} \cap \theta_{t_a} = \theta_{s_a \vee t_a} = \theta_{(s_a^a \wedge t_a^a)^a} = \theta_{(s' \wedge t')_a^a}$ and hence $f(\theta \cap \phi) = s' \wedge t' = f(\theta) \cap f(\phi)$. Similarly, we can prove that $f(\theta \vee \phi) = f(\theta) \vee f(\phi)$. Thus, $f$ is a homomorphism of Boolean algebras. Further, let $\theta, \phi \in \mathcal{B}(A)$ be as above such that $f(\theta) = f(\phi)$. Then $s' = t'$ and hence $s = t$ so that $s_a = t_a$ and $\theta \cap (A_a \times A_a) = \phi \cap (A_a \times A_a)$ for all $a \in A$. Now, we shall prove that $\theta = \phi$. Let $(x, y) \in \theta$. Then, for any $a \in A$

$(a\wedge x, a\wedge y) \in \theta \cap (A_a \times A_a) = \phi \cap (A_a \times A_a) \subseteq \phi$. Therefore $(a\wedge x, a\wedge y) \in \phi$ for all $a \in A$. In particular, $(x, x\wedge y)$ and $(y, y\wedge x) \in \phi \quad \rightarrow (1)$ and hence $(x\vee y, (x\wedge y)\vee(y\wedge x)) \in \phi$. By symmetry, $(y\vee x, (y\wedge x)\vee(x\wedge y)) \in \phi$. Since $(x\wedge y)\vee(y\wedge x) = (y\wedge x)\vee(x\wedge y)$, it follows that $(x\vee y, y\vee x) \in \phi. \quad \rightarrow (2)$ Also, since $(x', y') \in \theta$, we get that $((x\wedge y)', (y\wedge x)') = (y'\vee x', x'\vee y') \in \phi$ (by (2)) and hence $(x\wedge y, y\wedge x) \in \phi$. This and (1) gives that $(x, y) \in \phi$. Thus $\theta \subseteq \phi$. Similarly $\phi \subseteq \theta$. Thus $\theta = \phi$. Therefore $f$ is an injection too and hence $f$ is an embedding of $\mathcal{B}(A)$ into $\mathfrak{B}$. Thus $\mathcal{B}(A)$ is embedded in $\mathfrak{B}$.

**Corollary 2.14:** Let $A$ be a C-algebra with $T$. Then $\mathcal{B}(A)$, $\mathfrak{B}$ and $\mathbb{B}(A)$ are all isomorphic to each other.

**Proof:** In [7] it is proved that $\mathcal{B}(A)$ and $\mathbb{B}(A)$ are isomorphic to each other. We shall prove that the embedding $f : \mathcal{B}(A) \rightarrow \mathfrak{B}$, given in the proof of the above theorem, is a surjection too. Let $s \in \mathfrak{B}$. Then $s' \in \mathfrak{B}$ and $s'_a \in \mathbb{B}(A_a)$ for all $a \in A$ and $a \wedge s'_b = s'_a$ whenever $a \leq_* b$. In particular, since $a \leq_* T$, we have $a \wedge s'_T = s'_a$ for any $a \in A$. Now, $s'_T \in \mathbb{B}(A_T) = \mathbb{B}(A)$ (since $A_T = A$) and the congruence defined by $\theta = \theta_{s'_T}$ is a factor congruence on $A$ and $f(\theta) = s$; for, if $f(\theta) = t \in \mathfrak{B}$, then $\theta_{t'_a} = \theta \cap (A_a \times A_a) = \theta_{s'_T} \cap (A_a \times A_a) = \theta_{a\wedge s'_T} = \theta_{s'_a}$ and hence $t'_a = s'_a$ for all $a \in A$, so that $t = s$.Thus $f(\theta) = s$.Therefore $f$ is an isomorphism of $\mathcal{B}(A)$ onto $\mathfrak{B}$.

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