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On Non-Comaximal Graphs of Ideals of Commutative Rings

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Abstract. In this paper, we relate some properties of non-comaximal graph of ideals of a commutative ring with identity with the properties of the ring.

Key Words: Artinian Ring, Non-Comaximal Graph, Minimal Ideal
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Introduction

The relationship between an algebraic structure and a graph has expanded immensely after the introduction of a zero divisor graph by Istvan Beck [3] in 1988. Since then, several authors have defined many graphs such as comaximal graph of a commutative ring [13], intersection graphs of ideals of rings [4], the total graph of a commutative ring [1], etc. In [2], we introduced a graph associated with non-trivial (left) ideals of a ring, namely, a non-comaximal graph of ideals of a ring. The non-comaximal graph of ideals of a ring R, denoted by $NC(R)$, is an undirected graph whose vertex set is the collection of all non-trivial left ideals of R, and any two vertices are adjacent if and only if their sum is non-trivial in R. In this paper, we discuss some more properties of the non-comaximal graph of ideals of a commutative ring with unity. Throughout the paper, we use the results from [12], where a similar concept is discussed in module theory.

We recall some definitions and notations from graph and ring theories which are used below. Throughout this paper, all graphs are undirected. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The graph G is said to be empty if $E(G) = \emptyset$. We denote the degree of a vertex $v \in V(G)$ by $deg(v)$, that is the number of edges incident on v. If $deg(v) = 1$, then v is called an end vertex.

The graph G is complete if any two vertices of G are adjacent. A graph is said to be bipartite if its vertex set $V(G)$ can be partitioned into two subsets

V_1 and V_2 such that every edge of G joins V_1 and V_2. If $|V_1| = m$, $|V_2| = n$, and every vertex in V_1 (or V_2) is adjacent to all vertices in V_2 (or V_1), then the bipartite graph is said to be complete and is denoted by $K_{m,n}$. If either m or n is equal to 1, then $K_{m,n}$ is said to be a star.

A walk in G is an alternating sequence $v_0e_1v_1...e_nv_n$ of vertices and edges, in which each edge e_i is $v_{i-1}v_i$. A walk is said to be closed if it has the same first and last vertices. A path is a walk in which all vertices are distinct. We denote a path with n vertices by P_n. A circuit is a closed walk with all its vertices distinct (except the first and the last ones). The length of a circuit is the number of edges in the circuit. The length of the smallest circuit of G is called the girth of G, and is denoted by $girth(G)$. We say that G is connected if there is a path between every two distinct vertices, and G is disconnected if it is not connected.

If I and J are two distinct vertices of G, then $d(I,J)$ is the length of the shortest path from I to J, and if there is no such path, then $d(I,J) = \infty$. The diameter of G is the maximum distance among the distances between all pairs of vertices of G, and is denoted by $diam(G)$. A complete subgraph of G is said to be a clique in G. The number of vertices in maximum clique of G is called the clique number of G and is denoted by $\omega(G)$. By chromatic number $\chi(G)$ of G, we mean the minimum number of colors required to color the vertices in such a way that every two adjacent vertices have different colors.

In this paper, all rings are commutative with unity element. Let R be a ring. A non-zero ideal m of R is said to be minimal if it contains no other non-zero ideal. We use $min(R)$ to denote the set of minimal ideals of R. A simple ring is a non-zero ring that has no non-trivial proper ideal. Every minimal ideal is a simple ring. In a commutative ring R, R is simple if and only if R is a field. If I and J are two ideals of a ring R, then $\dfrac{I+J}{I} \cong \dfrac{J}{I \cap J}$.

A ring R is said to be Artinian if there does not exist an infinite strictly descending chain of ideals. A ring R is said to be Noetherian if there does not exist an infinite strictly ascending chain of ideals. In an Artinian ring, every ideal contains a minimal ideal. In a Noetherian ring, every ideal is contained in a maximal ideal. By $l(R)$, we denote the length of ascending/descending chain of R. A ring R is said to be local if it has exactly one maximal ideal. An ideal of a ring R is said to be small if it has a non-trivial sum with every non-trivial ideal of R.

Any undefined terminology can be found in [5–11].

1 Results

Now we present our main results.

Theorem 1 *The graph $NC(R)$ is disconnected if and only if R is a direct sum of two minimal ideals.*

Proof. Let $NC(R)$ be not connected. Suppose G_1, G_2 are two components of $NC(R)$ and $I \in G_1$, $J \in G_2$. Clearly, $I + J = R$, as there is no path between I and J. If $I \cap J \neq 0$, then $I + (I \cap J) = I$ and $J + (I \cap J) = J$, therefore $I - (I \cap J) - I$ is a path, which is a contradiction to the disconnectedness of $NC(R)$. Hence, $I \cap J = (0)$, this implies $R = I \bigoplus J$.

Assume that Z is an ideal of R such that $Z \subsetneq I$. Then $Z + I = I (\neq R)$, which implies that Z and I are adjacent vertices. Thus, $Z, I \in G_1$ which infers that there is no path between Z and J as $J \in G_2$. Therefore, $Z + J = R$. Now, $I = I \cap R = I \cap (Z + J) = Z + (I \cap J) = Z$. Hence, I is a minimal ideal of R. Similarly, J is also a minimal ideal of R. Conversely, assume that $R = F \bigoplus G$, where F and G are minimal ideals of R. Clearly, F and G are simple rings. As F is commutative, F is a field (see [10]). In the same way, G is also a field. Now, $\dfrac{R}{G} \cong F$ which infers that G is a maximal ideal of R. Similarly, F is also a maximal ideal of R. Assume that F is adjacent to N. Hence, $F + N \neq R$. Since F is maximal, $N \subseteq F$. The fact that F is minimal implies that $F = N$. Thus, F is an isolated vertex. □

Theorem 2 *If $NC(R)$ is connected, then the followings hold:*
(i) Every pair of maximal ideals of R has non-zero intersection, and they are connected.
(ii) Every pair of minimal ideals has non-trivial sum, and they are connected.

Proof. (i) Let F and G be two maximal ideals of R. Suppose $F \cap G = 0$. As F and G are maximal, this implies that $F + G = R$. Therefore, $F \bigoplus G = R$. Since $\dfrac{R}{F} \cong G$ and $\dfrac{R}{G} \cong F$, we obtain that F and G are fields. Hence, F and G are minimal ideals of R, which is a contradiction to connectedness of $NC(R)$. Therefore, $F \cap G \neq 0$. Also, $F - (F \cap G) - G$ is a path in $NC(R)$. This completes the proof.

(ii) Let f and g be two minimal ideals of R. Clearly, $f \cap g = 0$. If $f + g = R$, then $NC(R)$ is disconnected by Theorem 1. Therefore, $f + g \neq R$. □

Remark 1 *If $NC(R)$ is connected, then the set of all maximal ideals of R forms an independent set, and the set of all minimal ideals of R forms a clique.*

Theorem 3 *Let R be an Artinian ring. If R contains a unique minimal ideal, then $NC(R)$ is connected.*

Proof. Suppose R is an Artinian ring with unique minimal ideal m. If $I, J \in V(NC(R))$ are any two ideals, then $m \subseteq I$ and $m \subseteq J$. Therefore, $I - m - J$ is a path, and hence, $NC(R)$ is connected. □

Remark 2 *The graph $NC(Z_{p^2q^2})$, where p, q are primes, is a connected graph. Thus, the converse of the above theorem is not true.*

Theorem 4 *If $V(NC(R)) \geq 2$ and $NC(R)$ is disconnected, then the followings hold:*
(i) NC(R) is empty.
(ii) $l(R) = 2$, where $l(R)$ denotes the length of composition series of R.

Proof. Let $NC(R)$ be disconnected. Then $R = F \bigoplus G$ where F and G are minimal ideals of R. Clearly, F and G are simple rings. As F and G are commutative, F and G are fields (see [10]). Also, $\frac{R}{F} \cong G$ and $\frac{R}{G} \cong F$. Therefore, F and G are maximal ideals of R. Suppose $I(\neq F, G)$ is any ideal of R. It is clear that $I \subset R = F \bigoplus G$ and $I \cap F = 0 = I \cap G$. Again, we obtain $I \bigoplus F = R$, which implies $\frac{R}{F} \cong I$ and $\frac{R}{I} \cong F$. Therefore, I is maximal as well as minimal ideal of R. Therefore, every non-trivial ideal of R is minimal and maximal at the same time. By Theorem 2.3 in [2], $NC(R)$ is empty. □

Theorem 5 *The graph $NC(R)$ is complete if and only if R is local.*

Proof. Suppose R is a local ring. Then R possesses a unique maximal ideal M. If $I, J \in V(NC(R))$, then $I + J \subseteq M(\neq R)$. Thus, any two ideals are adjacent, therefore $NC(R)$ is complete. Conversely, let $NC(R)$ be complete. Suppose M_1, M_2 are two maximal ideals of R. Then $M_1 + M_2 = R$. Hence, M_1 and M_2 are not adjacent, which is a contradiction. This concludes the proof. □

Corollary 1 *The graph $NC(R)$ is complete if and only if every ideal of R is small.*

Corollary 2 *If $|V(NC(R))| = n$, then $deg(I) = n - 1$ for any small ideal I of R.*

Theorem 6 *If $\omega(NC(R)) < \infty$, then the following statements hold:*
(i) $l(R) < \infty$.
(ii) $\omega(NC(R)) = 1$ if and only if $|VNC(R)| = 1$ or R is a direct sum of two minimal ideals of R.
(iii) If $\omega(NC(R)) > 1$, then the number of minimal ideals of R is finite.

Proof. (i) Let $I_1 \subset I_2 \subset I_3 \subset \ldots$ be an infinite ascending chain of ideals. For $p < q$, $I_p + I_q = I_q(\neq R)$. Similar statement holds for descending chain of ideals. Hence, the infinite chain forms an infinite clique, which contradicts the assumption $\omega(NC(R)) < \infty$. Therefore, $l(R) < \infty$.

(ii) The second statement is trivial.

(iii) If $\omega(NC(R)) > 1$, then by (ii), $|VNC(R)| \neq 1$, and R is not a direct sum of two minimal ideals of R. By Theorem 1, $NC(R)$ is connected. Hence, by Theorem 2, every pair of minimal ideals of R has a non-trivial sum. Consider the subgraph $m^* = \{I : I \; is \; \; a \; minimal \; ideal \; of \; R\}$ of $NC(R)$ generated by all minimal ideals of R. Clearly, m^* forms a clique, and this implies that $|m^*| = \omega(m^*) \leq \omega(NC(R)) < \infty$. Hence, the number of minimal ideals of R is finite. □

Theorem 7 *If $|V(NC(R))| \geq 2$, then the following statements are equivalent:*
(i) $NC(R)$ is a star graph.
(ii) $NC(R)$ is tree.
(iii) $\chi(NC(R))) = 2$.
(iv) $l(R) = 3$, R is an Artinian ring with unique minimal ideal, and all other ideals are maximal.

Proof. $(i) \implies (ii)$ and $(ii) \implies (iii)$ are clear.

$(iii) \implies (iv)$ Let $\chi(NC(R))) = 2$. Since $\omega(NC(R) \leq \chi(NC(R))$, by Theorem 6, $l(R) < \infty$. Thus, there does not exit any infinite chain in R, hence R is Artinian, which infers that there exists a minimal ideal m in R. We claim that m is the only minimal ideal of R. Let $n(\neq m)$ be another minimal ideal of R. If $m + n = R$, then it contradicts the assumption $\chi(NC(R))) = 2$. Also, if $m + n \neq R$, then $m - (m + n) - n - m$ is a cycle of length 3, which contradicts $\chi(NC(R))) = 2$. Therefore, minimal ideal of R is unique. Consider an ideal K of R wich is distinct from m. If K is not maximal, then there exists an ideal X such that $K \subsetneqq X \subsetneqq R$ and also $m \subsetneqq K, X$, as R is an Artinian ring with unique minimal ideal. This implies that $m - K - X - m$ is a cycle of length 3, which is a contradiction. Hence, K is maximal, and only possible composition series in R is $m \subsetneqq K \subsetneqq R$, which implies that $l(R) = 3$.

$(iv) \implies (i)$ It is clear that all maximal ideals which are connected to the unique minimal ideal of R are end vertices of $NC(R)$. Therefore, $NC(R)$ is a star graph. □

Theorem 8 *If $deg(I) < \infty$ for every ideal I of a ring R, then $l(R) < \infty$.*

Proof. Assume that R contains an infinite ascending chain of ideals $I_1 \subset I_2 \subset I_3 \subset \ldots$ Then $deg(I_1) = \infty$ as $I_1 + I_j = I_j(\neq R)$ for all j. Equivalently, this happens for an infinite descending chain, too. Thus, $l(R) < \infty$. □

Lemma 1 *If m is minimal and $m + M = R$, then M is maximal.*

Proof. Suppose, M is not maximal. Then there exists an ideal P of R such that $M \subsetneqq P \subsetneqq R$. Therefore, $0 \subseteq m \cap M \subseteq m \cap P \subseteq m$. Clearly, $m \cap M = m$ or $m \cap P = 0$. If $m \cap M = m$, then $m \subseteq M$. This gives that $R = m + M = M$, which is a contradiction. If $m \cap P = 0$, then $M = P$ since $P = P \cap R = P \cap (m + M) = M + (m \cap P) = M + 0 = M$. Therefore, M maximal. □

Theorem 9 *Let m be a minimal ideal of R and $deg(m) < \infty$. If $NC(R)$ is connected, then the following statements hold:*
(i) the number of minimal ideals of R is finite.
(ii) $\chi(NC(R))) < \infty$.

Proof. Let $min(R) = \{m_i : m_i \text{ is a minimal ideal of } R\}$. Clearly, $m \in min(R)$, and hence, $min(R) \neq \emptyset$. By Theorem 2, $m + m_i \neq R$ for every $m_i \in min(R)$. Thus, $|min(R)| \leq deg(m) + 1 < \infty$. Therefore, $min(R)$ is finite.

(ii) Let $\{P_i\}$ be the set of ideals of R which are not adjacent to m. Then $m + P_i = R$ for every i. By Lemma 1, P_i is maximal, which implies that no two distinct vertices of $\{P_i\}$ are adjacent. Hence, all the vertices belonging to the set $\{P_i\}$ can be coloured by one colour. Also, consider the vertex set $\{Q_i\}$ of all vertices which are adjacent to m. Since $deg(m) < \infty$, $|\{Q_i\}|$ is finite. Therefore, the total number of colours required to colour $NC(R)$ is finite, that is, $\chi(NC(R))) < \infty$. □

Theorem 10 *If $NC(R)$ has no 3-cycle, then every maximal ideal is either an isolated vertex or an end vertex.*

Proof. Let M be a maximal ideal which is neither an isolated vertex nor an end vertex. Then $deg(M) \geq 2$, which infers that there exist at least two ideals I, J of R such that $I + M \neq R$ and $J + M \neq R$. As M is maximal, $I \subsetneqq M$ and $J \subsetneqq M$. Therefore, $I + J \subsetneqq M$, which implies that $I - M - J - I$ is a 3-cycle in $NC(R)$, a contradiction. □

Theorem 11 *If M is an end vertex of $NC(R)$, then either M is a maximal ideal or a minimal ideal of R.*

Proof. Suppose M is an end vertex of $NC(R)$. Therefore, there exists an ideal I such that $M + I \neq R$. Since $deg(M) = 1$, hence $M + I = M$ or $M + I = I$. If $M + I = I$, then $M \subsetneqq I$. If there exists an ideal J such that $0 \subsetneqq J \subsetneqq M$, then $J + M = M$. This contradicts the fact that $deg(M) = 1$. Hence, M is a minimal ideal of R. On the other hand, $M + I = M$ gives $I \subsetneqq M$. If there exists an ideal P such that $M \subsetneqq P \subsetneqq R$, then $P + M = P$, which contradicts the fact that $deg(M) = 1$. Thus, M is a maximal ideal of R. The proof is complete. □

Theorem 12 *The graph $NC(R) \cong P_2$ if and only if R has only two non-trivial ideals of R, one of which is maximal and the other is minimal.*

Proof. Assume that $NC(R) \cong P_2$. Let $I, J \in V(NC(R))$ be two adjacent vertices. Clearly, $I + J \neq R$. Therefore, either $I + J = I$ or $I + J = J$. If $I + J = I$, then it implies that I is maximal ideal and J is minimal ideal of R. Similarly, the other case gives J is a maximal ideal and I is a minimal ideal of R. The converse statement is obvious. □

Theorem 13 *If $NC(R)$ is a path, then $NC(R) \cong P_2$ or P_3.*

Proof. Let $NC(R)$ be a path $I_1 I_2 I_3 \ldots I_n \ldots$ Since I_1 is an end vertex of $NC(R)$, it is either a maximal ideal or a minimal ideal of R.

First, consider the case when I_1 is a minimal ideal of R. If I_2 is maximal, then clearly $NC(R) \cong P_2$. If not, we go on increasing the vertex number. For $n = 3$, $I_2 + I_3 \neq R$ as there is a path between I_2 and I_3. Then, there are three possibilities. In the first case, we consider $I_2 + I_3 = I_1$, which implies that $I_2 \subsetneqq I_1$. In the case when $I_2 + I_3 = I_2$, we have $I_3 \subsetneqq I_3$. This gives $I_1 + I_3 \subsetneqq I_1 + I_2 \neq R$. In the last case, we take $I_2 + I_3 = I_3$, and thus $I_2 \subsetneqq I_3$. The first two cases lead to the contradiction. Therefore, for $n = 3$, we get a path with I_1 being a minimal ideal of R and $I_2 \subsetneqq I_3$. For $n = 4$, we have four cases which are $I_3 + I_4 = I_1$, $I_3 + I_4 = I_2$, $I_3 + I_4 = I_3$ and $I_3 + I_4 = I_4$. All the four cases lead to the contradiction. Therefore, $NC(R) \cong P_3$.

Now consider the case when I_1 is a maximal ideal of R. Here, $I_2 \subsetneqq I_1$ as $I_1 + I_2 \neq R$. For $n = 3$, we get three cases, which are $I_2 + I_3 = I_1$, $I_2 + I_3 = I_2$ and $I_2 + I_3 = I_3$. Again, the first two cases lead to the contradiction. The third case gives $I_2 \subsetneqq I_3$. Thus, for n=3, we have a path with the conditions $I_2 \subsetneqq I_1$ and $I_2 \subsetneqq I_3$. For n=4, we get the following cases: $I_3 + I_4 = I_1$, $I_3 + I_4 = I_2$, $I_3 + I_4 = I_3$ and $I_3 + I_4 = I_4$. All of them lead to the contradiction. Thus, $NC(R) \cong P_3$. □

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