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On some optimizations of trigonometric interpolation using Fourier discrete coefficients

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Abstract

We investigate convergence of the rational-trigonometric-polynomial interpolations which perform convergence acceleration of the classical trigonometric interpolation by sequential application of polynomial and rational corrections. Rational corrections contain unknown parameters which determination outlines the behavior of the interpolations in different frameworks. We consider approach for determination of the unknown parameters by minimization of the constants of the asymptotic errors. We perform theoretical and numerical analysis of such optimal interpolations.

Key Words: Convergence Acceleration, Trigonometric Interpolation, Rational Interpolation, Krylov-Lanczos Interpolation
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Introduction

We continue investigations started in [12], [13] and [14] where we considered convergence acceleration of the classical trigonometric interpolation

$$I_N(f;x) = \sum_{n=-N}^{N} \check{f}_n e^{i\pi n x}, \ \check{f}_n = \frac{1}{2N+1} \sum_{k=-N}^{N} f(x_k) e^{-i\pi n x_k}, \ x_k = \frac{2k}{2N+1}$$

via sequential application of polynomial and rational correction functions. Polynomial correction was representing the discontinuities in the function and some of its first q derivatives ("jumps"). The resultant interpolation $I_{N,q}(f)$ was known as the Krylov-Lanczos (KL-) interpolation (see [1], [3]-[8], and [10] with references therein). Additional acceleration of the KL-interpolation was achieved by application of rational (by $e^{i\pi x}$) correction functions along the ideas of the Fourier-Pade approximations ([2]). This interpolation $I_{N,q}^{p}(f)$ was known as the rational-trigonometric-polynomial (RTP-) interpolation where p is the order of denominator in rational correction (see [12]-[14]).

Rational corrections contain unknown parameters θ_k which determination is a crucial issue for realization of the RTP-interpolations. In [12] and [13] it is assumed that

$$\theta_k = \theta_{-k} = 1 - \frac{\tau_k}{N}, \ k = 1, \dots, p, \tag{1}$$

where the new parameters τ_k can be determined differently and are independent of N. Papers [12] and [13] investigate the pointwise convergence of the RTP-interpolations in the regions away from the endpoints $x = \pm 1$ and in the numerical experiments consider the case when τ_k are the roots of the associated Laguerre polynomials. In this paper we continue investigations of the RTP-interpolations with parameters θ_k as in (1). We derive exact constants of the asymptotic errors and determine the parameters τ_k to be optimal in the sense of the considered frameworks: pointwise convergence in the regions away from the endpoints and L_2-convergence on the entire interval. This RTP-interpolations we call as pointwise-minimal and L_2-minimal RTP-interpolations. Theoretical and numerical analysis outline the properties of such interpolations.

1 The Krylov-Lanczos interpolation

First we recap some details from [10] concerning the polynomial corrections.

Let $f \in C^{q-1}[-1,1]$. By $A_k(f)$ denote the jumps of f at the end points of the interval

$$A_k(f) = f^{(k)}(1) - f^{(k)}(-1), \, k = 0, \dots, q-1.$$

The polynomial correction method is based on the following representation of the interpolated function

$$f(x) = \sum_{k=0}^{q-1} A_k(f) B_k(x) + F(x), \tag{2}$$

where B_k are 2-periodic Bernoulli polynomials

$$B_0(x) = \frac{x}{2}, \ B_k(x) = \int B_{k-1}(x)dx, \ x \in [-1,1], \ \int_{-1}^{1} B_k(x)dx = 0$$

with the Fourier coefficients

$$B_n(k) = \frac{(-1)^{n+1}}{2(i\pi n)^{k+1}}, \ n \neq 0, \ B_0(k) = 0.$$

Function F is a 2-periodic and relatively smooth function on the real line ($F \in C^{q-1}(R)$) with the discrete Fourier coefficients

$$\check{F}_n = \check{f}_n - \sum_{k=0}^{q-1} A_k(f) \check{B}_n(k).$$

Approximation of F in (2) by the classical trigonometric interpolation leads to the Krylov-Lanczos (KL-) interpolation

$$I_{N,q}(f;x) = \sum_{k=0}^{q-1} A_k(f)B_k(x) + I_N(F,x)$$

with the error

$$r_{N,q}(f;x) = f(x) - I_{N,q}(f;x).$$

We need some theoretical and numerical analysis for further comparisons. The next theorem reveals asymptotic behavior of the KL-interpolation in the framework of the L_2-norm.

Theorem 1. *[10] Let $f \in C^q[-1,1]$ and $f^{(q)} \in AC[-1,1]$ for some $q \geq 1$. Then the following estimate holds*

$$\lim_{N\to\infty} N^{q+\frac{1}{2}} \|r_{N,q}(f)\|_{L_2} = |A_q(f)| c(q),$$

where

$$c(q) = \frac{1}{\sqrt{2}\,\pi^{q+1}} \left(\frac{2}{2q+1} + \int_{-1}^{1} \left| \sum_{s\neq 0} \frac{(-1)^s}{(2s+x)^{q+1}} \right|^2 dx \right)^{1/2}.$$

Table 1 presents the values of $c(q)$.

q	$q=1$	$q=2$	$q=3$	$q=4$	$q=5$	$q=6$	$q=7$
$c(q)$	0.084	0.019	0.0055	0.0015	$4.4 \cdot 10^{-4}$	$1.3 \cdot 10^{-4}$	$3.8 \cdot 10^{-5}$

Table 1: Numerical values of $c(q)$ from Theorem 1.

Theorems 2 and 3 describe the pointwise convergence of the KL-interpolation in the regions away from the endpoints $x = \pm 1$.

Denote

$$\phi_m = \sum_{s=-\infty}^{\infty} \frac{(-1)^s}{(2s+1)^m}.$$

Theorem 2. *[10] Let $q \geq 2$ be even, $f \in C^{q+1}[-1,1]$ and $f^{(q+1)} \in AC[-1,1]$. Then the following estimate holds for $|x| < 1$*

$$r_{N,q}(f;x) = A_q(f) \frac{(-1)^{N+\frac{q}{2}}}{2\pi^{q+1}N^{q+1}} \frac{\sin\frac{\pi x}{2}(2N+1)}{\cos\frac{\pi x}{2}} \phi_{q+1} + o(N^{-q-1}),\ N\to\infty.$$

Theorem 3. *[10] Let $q \geq 1$ be odd, $f \in C^{q+2}[-1,1]$ and $f^{(q+2)} \in AC[-1,1]$. Then the following estimate holds for $|x| < 1$*

$$\begin{aligned} r_{N,q}(f;x) &= A_q(f) \frac{(-1)^{N+\frac{q+1}{2}+1}(q+1)}{4\pi^{q+1}N^{q+2}} \frac{\sin\frac{\pi x}{2} \sin\frac{\pi x}{2}(2N+1)}{\cos^2\frac{\pi x}{2}} \phi_{q+2} \\ &+ A_{q+1}(f) \frac{(-1)^{N+\frac{q+1}{2}}}{2\pi^{q+2}N^{q+2}} \frac{\sin\frac{\pi x}{2}(2N+1)}{\cos\frac{\pi x}{2}} \phi_{q+2} + o(N^{-q-2}),\ N\to\infty. \end{aligned}$$

Now consider the following simple testing function that we use for numerical analysis

$$f(x) = \sin(x-1). \tag{3}$$

As we see from Theorems 2 and 3 the behavior of the KL-interpolation (and also the behavior of the RTP-interpolations as we will show below) is different for even and odd q and in numerical examples we show the results for both cases. Figures 1 and 2 show the behavior of $|r_{N,q}(f;x)|$ on the interval $[-0.7, 0.7]$ (left figures) and at the point $x = 1$ (right figures) for $N = 2048$ and $q = 3$ and $q = 4$, respectively.

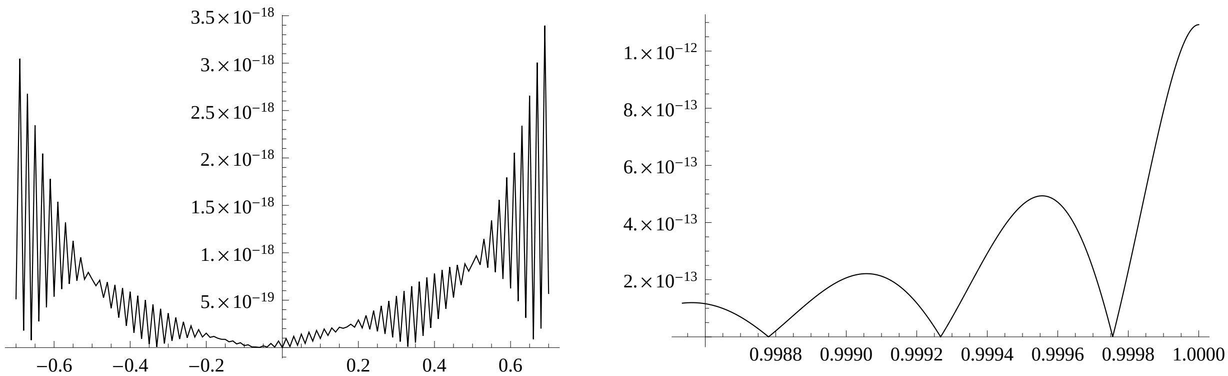

Figure 1: Graphs of $|r_{2048,3}(f;x)|$ on the interval $[-0.7, 0.7]$ (left) and at the point $x = 1$ (right) for the function (3).

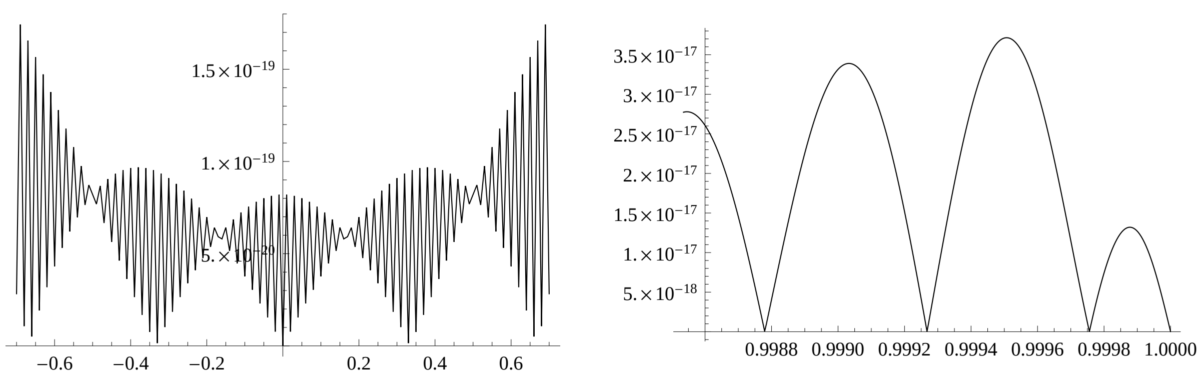

Figure 2: Graphs of $|r_{2048,4}(f;x)|$ on the interval $[-0.7, 0.7]$ (left) and at the point $x = 1$ (right) for the function (3).

We calculated also the L_2-norms of the errors

$$\|r_{2048,3}(f)\|_{L_2} = 2.0 \cdot 10^{-14}, \ \|r_{2048,4}(f)\|_{L_2} = 1.7 \cdot 10^{-18}. \tag{4}$$

2 Rational-trigonometric-polynomial interpolation

In this section we investigate the method of additional acceleration of convergence of the KL-interpolation by rational correction functions and recap the main ideas from [12] and

[13].

Consider a finite sequence of complex numbers $\theta = \{\theta_k\}_{|k|=1}^{p}$ and denote

$$\delta_n^0(\theta, c_n) = c_n,$$

$$\delta_n^k(\theta, c_n) = \delta_n^{k-1}(\theta, c_n) + \theta_{-k}\delta_{n-1}^{k-1}(\theta, c_n) + \theta_k\left(\delta_{n+1}^{k-1}(\theta, c_n) + \theta_{-k}\delta_n^{k-1}(\theta, c_n)\right)$$

for some sequence c_n. By $\delta_n^k(c_n)$ we denote the sequence that corresponds to the choice $\theta \equiv 1$. It is easy to check that

$$\delta_n^k(c_n) = \Delta_{n+k}^{2k}(c_n),$$

where $\Delta_n^k(c_n)$ are the classical backward finite differences defined by the recurrence relation

$$\begin{aligned} \Delta_n^0(c_n) &= c_n, \\ \Delta_n^k(c_n) &= \Delta_n^{k-1}(c_n) + \Delta_{n-1}^{k-1}(c_n). \end{aligned}$$

We can write according to expansion (2)

$$r_{N,q}(f;x) = \sum_{n=-N}^{N} (F_n - \check{F}_n)e^{i\pi nx} + \sum_{n=N+1}^{\infty} F_n e^{i\pi nx} + \sum_{n=-\infty}^{-N-1} F_n e^{i\pi nx},$$

where F_n is the n-th Fourier coefficient of F

$$F_n = \frac{1}{2}\int_{-1}^{1} F(x)e^{-i\pi nx}dx.$$

Now, we proceed by sequential applications of the Abel transformations and get

$$\begin{aligned} r_{N,q}(f;x) = (e^{-i\pi Nx} - e^{i\pi(N+1)x}) \sum_{k=1}^{p} \frac{\theta_{-k}\delta_N^{k-1}(\theta, \check{F}_n)}{\prod_{s=1}^{k}(1+\theta_{-s}e^{i\pi x})(1+\theta_s e^{-i\pi x})} \\ +(e^{i\pi Nx} - e^{-i\pi(N+1)x}) \sum_{k=1}^{p} \frac{\theta_k\delta_{-N}^{k-1}(\theta, \check{F}_n)}{\prod_{s=1}^{k}(1+\theta_{-s}e^{i\pi x})(1+\theta_s e^{-i\pi x})} \\ +\frac{1}{\prod_{s=1}^{p}(1+\theta_{-s}e^{i\pi x})(1+\theta_s e^{-i\pi x})} \sum_{|n|=N+1}^{\infty} \delta_n^p(\theta, F_n)e^{i\pi nx} \\ +\frac{1}{\prod_{s=1}^{p}(1+\theta_{-s}e^{i\pi x})(1+\theta_s e^{-i\pi x})} \sum_{n=-N}^{N} \delta_n^p(\theta, F_n - \check{F}_n)e^{i\pi nx}, \end{aligned}$$

which leads to the following rational-trigonometric-polynomial (RTP-) interpolation

$$\begin{aligned} I_{N,q}^p(f;x) &= I_{N,q}(f;x) + (e^{-i\pi Nx} - e^{i\pi(N+1)x}) \sum_{k=1}^{p} \frac{\theta_{-k}\delta_N^{k-1}(\theta, \check{F}_n)}{\prod_{s=1}^{k}(1+\theta_{-s}e^{i\pi x})(1+\theta_s e^{-i\pi x})} \\ &+ (e^{i\pi Nx} - e^{-i\pi(N+1)x}) \sum_{k=1}^{p} \frac{\theta_k\delta_{-N}^{k-1}(\theta, \check{F}_n)}{\prod_{s=1}^{k}(1+\theta_{-s}e^{i\pi x})(1+\theta_s e^{-i\pi x})} \end{aligned}$$

with the error

$$r_{N,q}^p(f;x) = \frac{1}{\prod_{s=1}^p(1+\theta_{-s}e^{i\pi x})(1+\theta_s e^{-i\pi x})}\sum_{|n|=N+1}^{\infty}\delta_n^p(\theta,F_n)e^{i\pi nx}$$
$$+ \frac{1}{\prod_{s=1}^p(1+\theta_{-s}e^{i\pi x})(1+\theta_s e^{-i\pi x})}\sum_{n=-N}^{N}\delta_n^p(\theta,F_n-\check{F}_n)e^{i\pi nx}. \qquad (5)$$

Throughout the paper we suppose that parameters θ_k are chosen as in (1). Determination of the new parameters τ_k will be discussed later. Let $\gamma_k(\tau)$ be the coefficients of the polynomial

$$\prod_{s=1}^{p}(1+\tau_s x) = \sum_{s=0}^{p}\gamma_s(\tau)x^s. \qquad (6)$$

Now, we investigate the convergence of the RTP-interpolation $I_{N,q}^p(f)$ in different frameworks. First we display some results from [13] where pointwise convergence in the regions away from the endpoints was explored. Then we consider L_2-convergence of the RTP-interpolations.

Denote

$$\psi_{m,p} = \sum_{s=0}^{p}(-1)^s\gamma_s(\tau)\sum_{k=0}^{p}\gamma_k(\tau)(2p-k-s+m)!\phi_{2p-k-s+m+1}.$$

Theorem 4. *[13] Let $q \ge 2$ be even and $f \in C^{q+2p+1}[-1,1]$ with $f^{(q+2p+1)} \in AC[-1,1]$ for some $p \ge 1$. Let parameters θ_k be chosen as in (1). Then the following estimate holds for $|x| < 1$*

$$r_{N,q}^p(f;x) = A_q(f)\frac{(-1)^{N+p+\frac{q}{2}}}{2^{2p+1}\pi^{q+1}q!N^{2p+q+1}}\frac{\sin\frac{\pi x}{2}(2N+1)}{\cos^{2p+1}\frac{\pi x}{2}}\psi_{q,p} + o(N^{-2p-q-1}),\ N\to\infty.$$

Theorem 5. *[13] Let $q \ge 1$ be odd and $f \in C^{q+2p+2}[-1,1]$ with $f^{(q+2p+2)} \in AC[-1,1]$ for some $p \ge 1$. Let parameters θ_k be chosen as in (1). Then the following estimate holds for $|x| < 1$*

$$r_{N,q}^p(f;x) = A_q(f)\frac{(-1)^{N+p+\frac{q+1}{2}+1}}{2^{2p+2}\pi^{q+1}q!N^{2p+q+2}}\frac{\sin\frac{\pi x}{2}\sin\frac{\pi x}{2}(2N+1)}{\cos^{2p+2}\frac{\pi x}{2}}\psi_{q+1,p}$$
$$+ A_{q+1}(f)\frac{(-1)^{N+p+\frac{q+1}{2}}}{2^{2p+1}\pi^{q+2}(q+1)!N^{2p+q+2}}\frac{\sin\frac{\pi x}{2}(2N+1)}{\cos^{2p+1}\frac{\pi x}{2}}\psi_{q+1,p}$$
$$+ o(N^{-2p-q-2}),\ N\to\infty.$$

Comparison with Theorems 2 and 3 shows that for smooth functions RTP-interpolations with θ as in (1) are asymptotically more precise than the KL-interpolation and improvement in precision is by the factor $O(N^{2p})$ as $N\to\infty$. Also worth noting that interpolation by odd q has more accuracy although its asymptotic depends not only on $A_q(f)$ but also on $A_{q+1}(f)$.

Next theorem describes the asymptotic behavior of the error of the RTP-interpolations in the L_2-norm.

Theorem 6. *Let $f \in C^{q+2p}[-1,1]$ with $f^{(q+2p)} \in AC[-1,1]$ for some $p, q \geq 1$. Let*

$$\theta_k = \theta_{-k} = 1 - \frac{\tau_k}{N}, \ \tau_k > 0, \ \tau_i \neq \tau_j, \ i \neq j.$$

Then the following estimate holds

$$\lim_{N\to\infty} N^{q+\frac{1}{2}} \|r_{N,q}^p(f)\|_{L_2} = |A_q(f)| c_p(q),$$

where

$$\begin{aligned} c_p(q) &= \frac{1}{2\pi^{q+1} q!} \\ &\times \Bigg(\frac{1}{2} \int_{-1}^{1} \Bigg| \int_{1}^{\infty} (a(x-t) + (-1)^{q+1} a(t+x)) \Omega(x) dx \\ &\qquad - \int_{t}^{1} a(x-t) \Upsilon(x) dx - \int_{-1}^{t} a(t-x) \Upsilon(x) dx \Bigg|^2 dt \\ &+ \int_{1}^{\infty} \Bigg| \int_{1}^{t} (a(t-x) + (-1)^{q+1} a(t+x)) \Omega(x) dx \\ &\qquad + \int_{t}^{\infty} (a(x-t) + (-1)^{q+1} a(t+x)) \Omega(x) dx \\ &\qquad - \int_{-1}^{1} a(t-x) \Upsilon(x) dx \Bigg|^2 dt \Bigg)^{\frac{1}{2}}, \end{aligned}$$

and

$$\Omega(x) = \frac{1}{x^{q+2p+1}} \sum_{s=0}^{p} (-1)^s \gamma_s(\tau) x^s \sum_{k=0}^{p} \gamma_k(\tau) x^k (2p-k-s+q)!,$$

$$\Upsilon(x) = \sum_{s=0}^{p} (-1)^s \gamma_s(\tau) \sum_{k=0}^{p} \gamma_k(\tau) (2p-k-s+q)! \sum_{r\neq 0} \frac{(-1)^r}{(2r+x)^{2p-k-s+q+1}},$$

$$a(x) = \sum_{s=1}^{p} \frac{e^{-\tau_s x}}{\tau_s \prod_{\substack{r=1 \\ r\neq s}}^{p} (\tau_r^2 - \tau_s^2)}.$$

Proof. We use (5) and proceed by estimation of the following fraction

$$\frac{1}{\prod_{s=1}^{p}(1+\theta_s e^{i\pi x})(1+\theta_s e^{-i\pi x})} = \sum_{s=1}^{p} \frac{\beta_s}{(1+\theta_s e^{i\pi x})(1+\theta_s e^{-i\pi x})},$$

where

$$\beta_m = \frac{1}{\prod_{\substack{r=1 \\ r\neq m}}^{p} (1-\theta_r\theta_m)\left(1 - \frac{\theta_r}{\theta_m}\right)} = \frac{\theta_m^{p-1}}{\prod_{\substack{r=1 \\ r\neq m}}^{p} (1-\theta_r\theta_m)(\theta_m - \theta_r)}.$$

Taking into account that parameters θ_k are chosen as in (1) we derive

$$\beta_m = \frac{\left(1-\frac{\tau_m}{N}\right)^{p-1} N^{2p-2}}{\prod\limits_{\substack{r=1\\ r\neq m}}^{p} (\tau_r-\tau_m)\left(\tau_r+\tau_m-\frac{\tau_r\tau_m}{N}\right)}$$

and hence

$$\frac{1}{N^{2p-2}}\lim_{N\to\infty}\beta_m = \frac{1}{\prod\limits_{\substack{r=1\\ r\neq m}}^{p}(\tau_r^2-\tau_m^2)}. \tag{7}$$

Then, as $|\theta_k|<1$, however for sufficient large N, we can write

$$\begin{aligned}
\sum_{s=1}^{p}\frac{\beta_s}{(1+\theta_s e^{i\pi x})(1+\theta_s e^{-i\pi x})} &= \sum_{s=1}^{p}\beta_s\sum_{n=0}^{\infty}(-1)^n\theta_s^n e^{i\pi n x}\sum_{m=0}^{\infty}(-1)^m\theta_s^m e^{-i\pi m x}\\
= \sum_{s=1}^{p}\beta_s\sum_{\ell=-\infty}^{-1}&(-1)^\ell e^{i\pi\ell x}\theta_s^{-\ell}\sum_{n=0}^{\infty}\theta_s^{2n}+\sum_{s=1}^{p}\beta_s\sum_{\ell=0}^{\infty}(-1)^\ell e^{i\pi\ell x}\theta_s^{\ell}\sum_{n=0}^{\infty}\theta_s^{2n}\\
= \sum_{s=1}^{p}\frac{\beta_s}{1-\theta_s^2}&\Bigg[\sum_{\ell=-\infty}^{-1}(-1)^\ell e^{i\pi\ell x}\theta_s^{-\ell}+\sum_{\ell=0}^{\infty}(-1)^\ell e^{i\pi\ell x}\theta_s^{\ell}\Bigg],
\end{aligned} \tag{8}$$

where according to (7) and (1)

$$\frac{1}{N^{2p-1}}\lim_{N\to\infty}\frac{\beta_s}{1-\theta_s^2} = \frac{1}{2\tau_s\prod\limits_{\substack{r=1\\ r\neq s}}^{p}(\tau_r^2-\tau_s^2)}. \tag{9}$$

Substituting (8) into (5), after simple manipulations we obtain

$$r_{N,q}^p(f) = I_1+I_2+I_3,$$

where

$$\begin{aligned}
I_1 = &\sum_{t=N+1}^{\infty} e^{i\pi t x}\sum_{s=1}^{p}\frac{\beta_s}{1-\theta_s^2}\Bigg(\sum_{n=N+1}^{t}(-1)^{t+n}\theta_s^{t-n}\delta_n^p(\theta,F_n)\\
&+\sum_{n=t+1}^{\infty}(-1)^{t+n}\theta_s^{n-t}\delta_n^p(\theta,F_n)+\sum_{n=-\infty}^{-N-1}(-1)^{t+n}\theta_s^{t-n}\delta_n^p(\theta,F_n)\\
&+\sum_{n=-N}^{N}(-1)^{t+n}\theta_s^{t-n}\delta_n^p(\theta,F_n-\check{F}_n)\Bigg),
\end{aligned}$$

then

$$\begin{aligned} I_2 &= \sum_{t=-N}^{N} e^{i\pi tx} \sum_{s=1}^{p} \frac{\beta_s}{1-\theta_s^2} \Bigg(\sum_{n=N+1}^{\infty} (-1)^{t+n} \theta_s^{n-t} \delta_n^p(\theta, F_n) \\ &+ \sum_{n=-\infty}^{-N-1} (-1)^{t+n} \theta_s^{t-n} \delta_n^p(\theta, F_n) + \sum_{n=t+1}^{N} (-1)^{t+n} \theta_s^{n-t} \delta_n^p(\theta, F_n - \check{F}_n) \\ &+ \sum_{n=-N}^{t} (-1)^{t+n} \theta_s^{t-n} \delta_n^p(\theta, F_n - \check{F}_n) \Bigg), \end{aligned}$$

and

$$\begin{aligned} I_3 &= \sum_{t=-\infty}^{-N-1} e^{i\pi tx} \sum_{s=1}^{p} \frac{\beta_s}{1-\theta_s^2} \Bigg(\sum_{n=N+1}^{\infty} (-1)^{t+n} \theta_s^{n-t} \delta_n^p(\theta, F_n) \\ &+ \sum_{n=t+1}^{-N-1} (-1)^{t+n} \theta_s^{n-t} \delta_n^p(\theta, F_n) + \sum_{n=-\infty}^{t} (-1)^{t+n} \theta_s^{t-n} \delta_n^p(\theta, F_n) \\ &+ \sum_{n=-N}^{N} (-1)^{t+n} \theta_s^{n-t} \delta_n^p(\theta, F_n - \check{F}_n) \Bigg). \end{aligned}$$

Hence

$$\|r_{N,q}^p(f)\|_{L_2}^2 = \|I_1\|_{L_2}^2 + \|I_2\|_{L_2}^2 + \|I_3\|_{L_2}^2, \tag{10}$$

where

$$\begin{aligned} \|I_1\|_{L_2}^2 &= 2 \sum_{t=N+1}^{\infty} \Bigg| \sum_{s=1}^{p} \frac{\beta_s}{1-\theta_s^2} \sum_{n=N+1}^{t} (-1)^{t+n} \theta_s^{t-n} \delta_n^p(\theta, F_n) \\ &+ \sum_{n=t+1}^{\infty} (-1)^{t+n} \theta_s^{n-t} \delta_n^p(\theta, F_n) + \sum_{n=-\infty}^{-N-1} (-1)^{t+n} \theta_s^{t-n} \delta_n^p(\theta, F_n) \\ &+ \sum_{n=-N}^{N} (-1)^{t+n} \theta_s^{t-n} \delta_n^p(\theta, F_n - \check{F}_n) \Bigg|^2 \end{aligned}$$

and similar formulas we have for $\|I_2\|_{L_2}^2$ and $\|I_3\|_{L_2}^2$.

Now we estimate $\delta_n^p(\theta, F_n)$ and $\delta_n^p(\theta, \check{F}_n - F_n)$. According to smoothness of f and expansion (2) we write

$$F_n = \sum_{m=q}^{q+2p} A_m(f) B_n(m) + o(n^{-2p-q-1})$$

and

$$\delta_n^p(\theta, F_n) = \sum_{m=q}^{q+2p} A_m(f) \delta_n^p(\theta, B_n(m)) + o(n^{-q-2p-1}).$$

Application of Lemma 1 leads to the following estimate

$$\begin{aligned}\delta_n^p(\theta, F_n) &= A_q(f)\frac{(-1)^{n+p+1}}{2(i\pi n)^{q+1}n^{2p}q!}\sum_{s=0}^{p}(-1)^s\frac{\gamma_s(\tau)}{N^s n^{-s}}\sum_{k=0}^{p}\frac{\gamma_k(\tau)}{N^k n^{-k}}(2p-k-s+q)! \\ &+ \frac{1}{N^{2p}}o(n^{-q-1}),\ n \geq N+1,\ N \to \infty.\end{aligned}$$

Then, again according to expansion (2) and smoothness of f we have

$$\delta_n^p(\theta, \check{F}_n - F_n) = \sum_{m=q}^{q+2p} A_m(f)\delta_n^p(\theta, \check{B}_n(m) - B_n(m)) + o(N^{-2p-q-1}).$$

Now application of Lemma 2 leads to the following estimate

$$\begin{aligned}\delta_n^p(\theta, \check{F}_n - F_n) &= A_q(f)\frac{(-1)^{n+p+1}}{2(i\pi N)^{q+1}N^{2p}q!}\sum_{s=0}^{p}(-1)^s\frac{\gamma_s(\tau)}{N^s}\sum_{k=0}^{p}\frac{\gamma_k(\tau)}{N^k}(2p-k-s+q)! \\ &\times \sum_{r\neq 0}\frac{(-1)^r}{\left(2r+\frac{n}{N}\right)^{2p-k-s+q+1}} + o(N^{-2p-q-1}),\ |n| \leq N,\ N \to \infty.\end{aligned}$$

These complete the proof by tending N to infinity in (10) and by replacing the sums by the corresponding integrals and by taking into account (9). □

Theorems 4, 5, 6 are valid nonetheless parameters τ_k are still undefined. Papers [12] and [13] consider parameters τ_k which are the roots of the associated Laguerre polynomials $L_p^q(x)$

$$L_p^q(\tau_k) = 0,\ k = 1, \ldots, p.$$

It is well-known that the roots are distinct and positive. Associated Laguerre polynomials have well-known representation

$$L_p^q(x) = \sum_{k=0}^{p}(-1)^k\frac{(p+q)!}{k!(p-k)!(q+k)!}x^k.$$

It allows calculation of parameters τ_k explicitly for $p = 1, 2, 3$. For other values of p the values of τ_k can be calculated numerically with any required precision.

Table 2 displays the values of $c_p(q)$ when τ_k are the roots of the associated Laguerre polynomials $L_p^q(x)$.

The ratio $c(q)/c_p(q)$ shows the efficiency of the RTP-interpolation compared to the KL-interpolation. We see that as higher are the values of p and q as more efficient is the RTP-interpolation (in general).

One thing that worth to notice is that Theorems 4, 5 and 6 put additional smoothness requirements on the interpolated function compared to Theorems 1, 2 and 3 so in comparisons this fact must be taken into account.

q	1	2	3	4	5
$c_1(q)$	0.020	0.0031	0.00092	0.00016	0.000047
$c(q)/c_1(q)$	4.1	6.2	6.1	9.5	9.4
$c_2(q)$	0.0091	0.0014	0.00027	0.000038	0.000012
$c(q)/c_2(q)$	9.2	13.7	20.6	39.8	36.0
$c_3(q)$	0.0064	0.0007	0.00009	0.000017	$3.9 \cdot 10^{-6}$
$c(q)/c_3(q)$	13.1	27.3	61.8	91.5	115.2
$c_4(q)$	0.0046	0.00038	0.000045	$8.0 \cdot 10^{-6}$	$1.3 \cdot 10^{-6}$
$c(q)/c_4(q)$	18.3	50.0	123.3	190.9	351.9

Table 2: Numerical values of $c_p(q)$ and $c(q)/c_p(q)$ when parameters τ_k are the roots of $L_p^q(x)$.

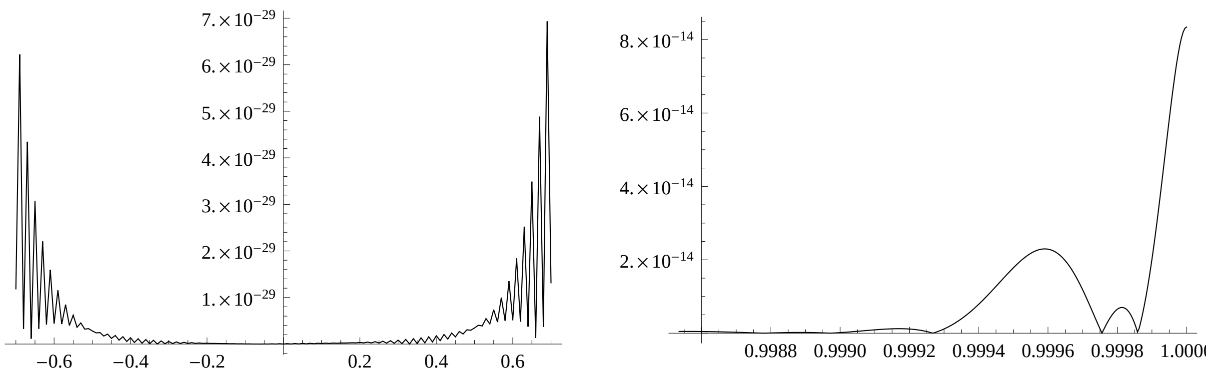

Figure 3: Graphs of $|r^2_{2048,3}(f;x)|$ on the interval $[-0.7, 0.7]$ (left) and at the point $x = 1$ (right) for the function (3) when parameters τ_k are the roots of $L_2^3(x)$.

Figures 3 and 4 show the behavior of $|r^p_{N,q}(f;x)|$ on the interval $[-0.7, 0.7]$ (left figures) and at the point $x = 1$ (right figures) for $p = 2$, $N = 2048$ and $q = 3$ and $q = 4$, respectively, when parameters τ_k are the roots of $L_p^q(x)$.

We have the following L_2-errors

$$\|r^2_{2048,3}(f)\|_{L_2} = 9.7 \cdot 10^{-16}, \ \|r^2_{2048,3}(f)\|_{L_2} = 4.4 \cdot 10^{-20}. \tag{11}$$

Comparison with Figures 1, 2 and (4) shows tremendous improvement in accuracy both by L_2 and pointwise convergence.

3 Optimal RTP-interpolations

As was mentioned above in Theorems 4, 5 and 6 parameters τ_k are undetermined and this gives opportunity to achieve additional accuracy in different frameworks by minimization of the constants in the asymptotic errors. Minimization of $c_p(q)$ in Theorem 6 leads to L_2-

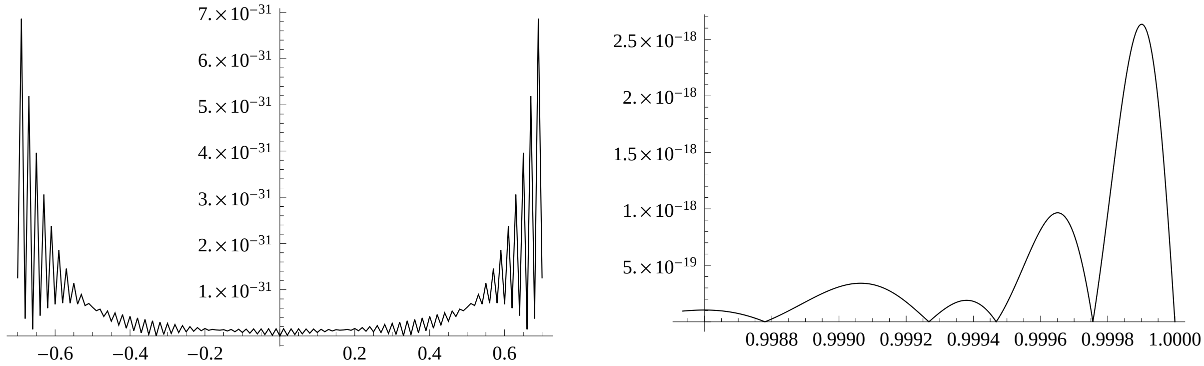

Figure 4: Graphs of $|r^2_{2048,4}(f;x)|$ on the interval $[-0.7, 0.7]$ (left) and at the point $x = 1$ (right) for the function (3) when parameters τ_k are the roots of $L^4_2(x)$.

minimal RTP-interpolation and minimization of $\psi_{q,p}$ and $\psi_{q+1,p}$ in Theorems 4 and 5 leads to pointwise (P-) minimal RTP-interpolation.

3.1 L_2-minimal RTP-interpolations

The idea of determination of parameters τ_k by minimization of the constant $c_p(q)$ in the estimate of the L_2-error was realized in [9] for RTP-approximations. For interpolations this idea was realized in [11] only for $p = 1$. Method described there was not allowing to get parameters for other values of p while estimate of Theorem 6 is giving such possibility. Thus, we determine parameters τ_k from the condition

$$c_p(q) \to minimum. \tag{12}$$

Tables 3 and 4 show the values of τ_k that solve the problem (12) for $p = 1$ and $p = 2$, respectively.

q	1	2	3	4	5
τ_1	1.8081	2.4581	3.7303	4.3705	5.7525
$c_1(q)$	0.015	0.0022	0.00070	0.000084	0.000038
$c(q)/c_1(q)$	5.5	8.8	7.9	18.1	11.8

Table 3: Numerical values of τ_1, $c_1(q)$ for the L_2-minimal RTP-interpolation.

As it was expected the L_2-minimal RTP-interpolation is more accurate not only compared to the KL-interpolation but also compared to the RTP-interpolation by the roots of the associated Laguerre polynomials. For example, when $p = 2$ and $q = 5$ the L_2-minimal RTP-interpolation is 105 times more accurate (asymptotically) in the L_2-norm than the KL-interpolation while RTP-interpolation by the roots of the Laguerre polynomial is only 36 times more accurate.

q	1	2	3	4	5
τ_1	0.7737	1.3199	2.2877	2.6571	3.7031
τ_2	3.8711	4.5984	6.5213	6.9081	8.7884
$c_2(q)$	0.0051	0.00041	0.000096	0.000015	$4.2 \cdot 10^{-6}$
$c(q)/c_2(q)$	16.6	46.0	57.5	102.8	104.6

Table 4: Numerical values of τ_1, τ_2 and $c_2(q)$ for the L_2-minimal RTP-interpolation.

Figures 5 and 6 show the behavior of the L_2-minimal RTP-interpolation $|r^p_{N,q}(f;x)|$ on the interval $[-0.7, 0.7]$ (left figures) and at the point $x = 1$ (right figures) for $p = 2$, $N = 2048$ and $q = 3$ and $q = 4$, respectively.

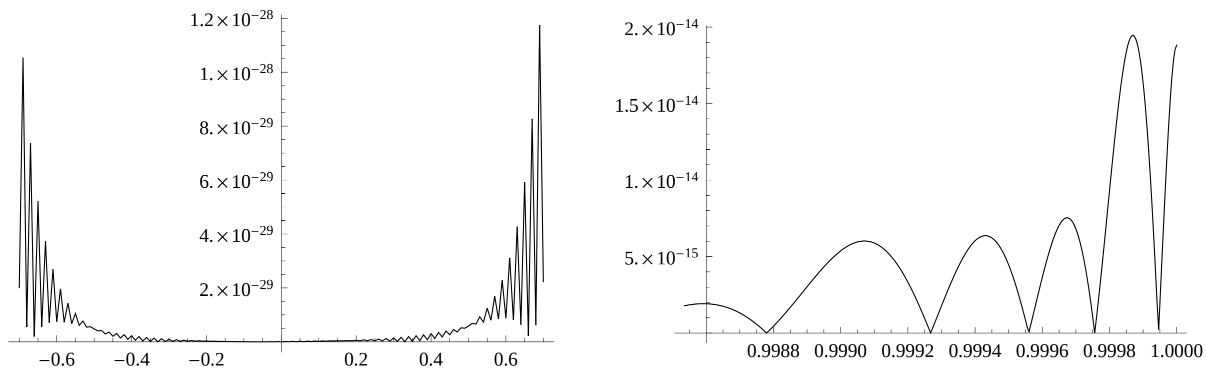

Figure 5: Graphs of $|r^p_{2048,q}(f;x)|$ on the interval $[-0.7, 0.7]$ (left) and at the point $x = 1$ (right) when function (3) is interpolated by the L_2-minimal interpolation with $p = 2$ and $q = 3$.

Comparison with Figures 3 and 4 shows that the L_2-minimal interpolation is more precise on the entire interval by the L_2 and uniform norms (compare also (13) with (11)) but less accurate in the regions away from the endpoints.

However, while applying the L_2-minimal interpolation we have serious limitation - optimal values can be calculated only for limited values of p and q as minimization of $c_p(q)$ is not an easy problem while calculation of the roots of the Laguerre polynomials can be performed actually for rather large values of p and q with actually any required precision.

We have the following L_2-errors for the L_2-minimal interpolations

$$\|r^2_{2048,3}(f)\|_{L_2} = 3.6 \cdot 10^{-16}, \ \|r^2_{2048,3}(f)\|_{L_2} = 1.7 \cdot 10^{-20}. \tag{13}$$

3.2 Pointwise minimal RTP-interpolations

In this subsection we investigate determination of parameters τ_k that leads to RTP-interpolations with more accuracy in the regions away from the endpoints than RTP-interpolations introduced above. The resultant RTP-interpolation we call as pointwise (P-) minimal RTP-interpolation.

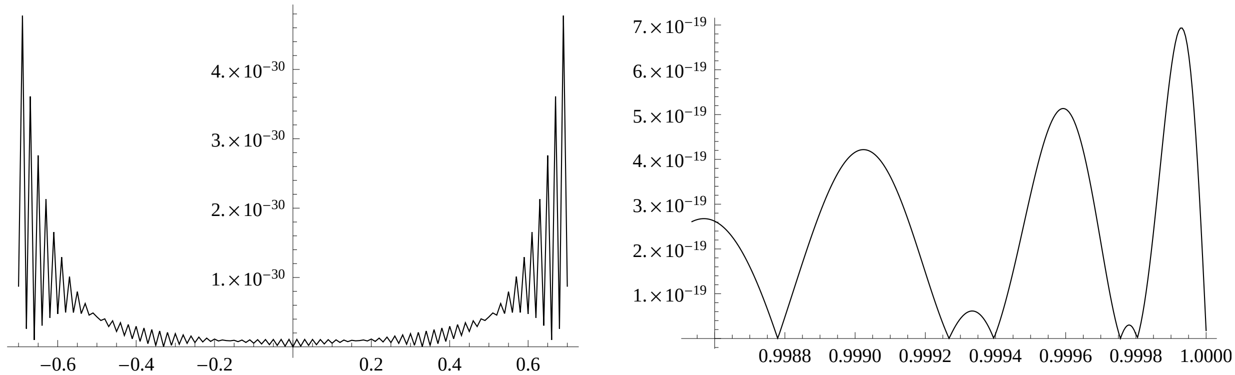

Figure 6: Graphs of $|r^p_{2048,q}(f;x)|$ on the interval $[-0.7, 0.7]$ (left) and at the point $x = 1$ (right) when function (3) is interpolated by the L_2-minimal interpolation with $p = 2$ and $q = 4$.

Let us consider the error of the RTP-interpolation given by (5). We derive by sequential applications of the Abel transformations the following expansion of the error when $|x| < 1$

$$\begin{aligned} r^p_{N,q}(f;x) &= \frac{e^{-i\pi Nx} - e^{i\pi(N+1)x}}{\prod_{s=1}^p (1+\theta_{-s}e^{i\pi x})(1+\theta_s e^{-i\pi x})} \sum_{w=0}^{p-1} \frac{\delta^w_N(\delta^p_n(\theta, \check{F}_n))}{(1+e^{i\pi x})^{w+1}(1+e^{-i\pi x})^{w+1}} \\ &+ \frac{e^{i\pi Nx} - e^{-i\pi(N+1)x}}{\prod_{s=1}^p (1+\theta_{-s}e^{i\pi x})(1+\theta_s e^{-i\pi x})} \sum_{w=0}^{p-1} \frac{\delta^w_{-N}(\delta^p_n(\theta, \check{F}_n))}{(1+e^{i\pi x})^{w+1}(1+e^{-i\pi x})^{w+1}} \\ &+ \frac{1}{(2+2\cos\pi x)^p \prod_{s=1}^p (1+\theta_{-s}e^{i\pi x})(1+\theta_s e^{-i\pi x})} \sum_{|n|=N+1}^{\infty} \delta^p_n(\delta^p_n(\theta, F_n))e^{i\pi nx} \\ &+ \frac{1}{(2+2\cos\pi x)^p \prod_{s=1}^p (1+\theta_{-s}e^{i\pi x})(1+\theta_s e^{-i\pi x})} \sum_{n=-N}^{N} \delta^p_n(\delta^p_n(\theta, F_n - \check{F}_n))e^{i\pi nx}. \end{aligned}$$

From here it turns out that for better accuracy in the regions away from the endpoints $x = \pm 1$ parameters τ_k must be determined from the conditions

$$\delta^w_N(\delta^p_n(\theta, \check{F}_n)) = \delta^w_{-N}(\delta^p_n(\theta, \check{F}_n)) = 0, \ w = 0, \dots, p-1$$

and taking into account the asymptotic expansions for $\delta^w_{\pm N}(\delta^p_n(\theta, \check{B}_n(m)))$ in Lemmas 3 and 4 we get the following system of equations for determination of γ_k and hence from (6) for determination of τ_k

$$\psi_{q+w,p} = 0, \ w = 0, \dots, p-1 \tag{14}$$

for even values of q, and

$$\sum_{s=0}^{p} (-1)^s \gamma_s(\tau) \sum_{k=0}^{p} \gamma_k(\tau)(2w+2p-k-s+m+1)! \sum_{r=-\infty}^{\infty} \frac{(-1)^r r}{(2r+1)^{2w+2p-k-s+m+2}} = 0,$$
$$w = 0, \dots, p-1$$

for odd q. The latest is equivalent to the following system of equations

$$\psi_{q+w+1,p} = 0, \ w = 0, \dots, p-1. \tag{15}$$

Now, estimates of Theorems 4 and 5 imply.

Theorem 7. *Let $q \geq 2$ be even and $f \in C^{q+2p+1}[-1,1]$ with $f^{(q+2p+1)} \in AC[-1,1]$ for some $p \geq 1$. Let parameters θ_k be chosen as in (1) where coefficients $\gamma_k(\tau)$ in (6) satisfy the system (14). Then the following estimate holds for $|x| < 1$*

$$r_{N,q}^{p}(f;x) = o(N^{-2p-q-1}), \ N \to \infty.$$

Theorem 8. *Let $q \geq 1$ be odd and $f \in C^{q+2p+2}[-1,1]$ with $f^{(q+2p+2)} \in AC[-1,1]$ for some $p \geq 1$. Let parameters θ_k be chosen as in (1) where coefficients $\gamma_k(\tau)$ in (6) satisfy the system (15). Then the following estimate holds for $|x| < 1$*

$$r_{N,q}^{p}(f;x) = o(N^{-2p-q-2}), \ N \to \infty.$$

Tables 5 and 6 displays the values of τ_k calculated from systems (14) and (15) together with the values $c_p(q)$ and $c(q)/c_p(q)$. Table 5 shows the values for $p = 1$ and Table 6 the values for $p = 2$. Similarly other values of p can be investigated. Comparison with the above similar tables for other RTP-interpolations shows that P-minimal RTP-interpolation has the worst accuracy in the L_2-norm.

q	1	2	3	4	5
τ_1	3.5124	3.5124	5.4866	5.4866	7.4848
$c_1(q)$	0.059	0.0044	0.0027	0.00024	0.000016
$c(q)/c_1(q)$	1.4	4.3	2.1	6.3	2.8

Table 5: Numerical values of τ_1, $c_1(q)$ and $c(q)/c_1(q)$ for the P-minimal RTP-interpolation.

q	1	2	3	4	5
τ_1	2.8699	2.8699	4.4990	4.4990	6.1829
τ_2	8.1108	8.1108	10.4512	10.4512	12.7675
$c_2(q)$	0.050	0.0030	0.0019	0.00012	0.000093
$c(q)/c_2(q)$	1.67	6.2	2.9	12.3	4.8

Table 6: Numerical values of τ_1, τ_2, $c_2(q)$ and $c(q)/c_2(q)$ for the P-minimal RTP-interpolation.

Figures 7 and 8 show the behavior of the P-minimal RTP-interpolation $|r_{N,q}^{p}(f;x)|$ on the interval $[-0.7, 0.7]$ (left figures) and at the point $x = 1$ (right figures) for $p = 2$, $N = 2048$ and $q = 3$ and $q = 4$, respectively. Comparison with the figures presented above shows the worst L_2 and uniform accuracy on the entire interval but the best pointwise accuracy in the regions away from the endpoints (compare also (16) with the above norms of the L_2-errors).

We have the following L_2-errors for the P-minimal interpolations

$$\|r_{2048,3}^{2}(f)\|_{L_2} = 6.9 \cdot 10^{-15}, \ \|r_{2048,3}^{2}(f)\|_{L_2} = 1.4 \cdot 10^{-19}. \qquad (16)$$

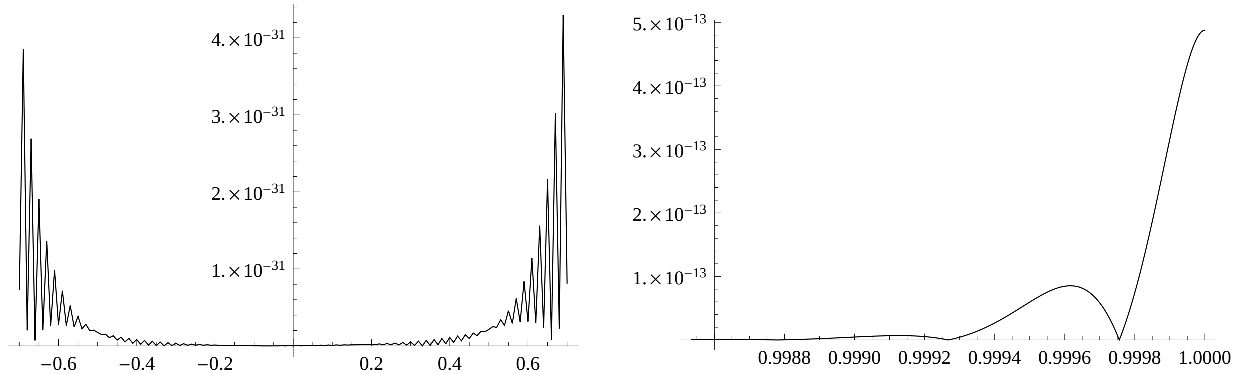

Figure 7: Graphs of $|r^p_{2048,q}(f;x)|$ on the interval $[-0.7, 0.7]$ (left) and at the point $x = 1$ (right) for function (3) when parameters τ_k are determined from (15) with $p = 2$ and $q = 3$.

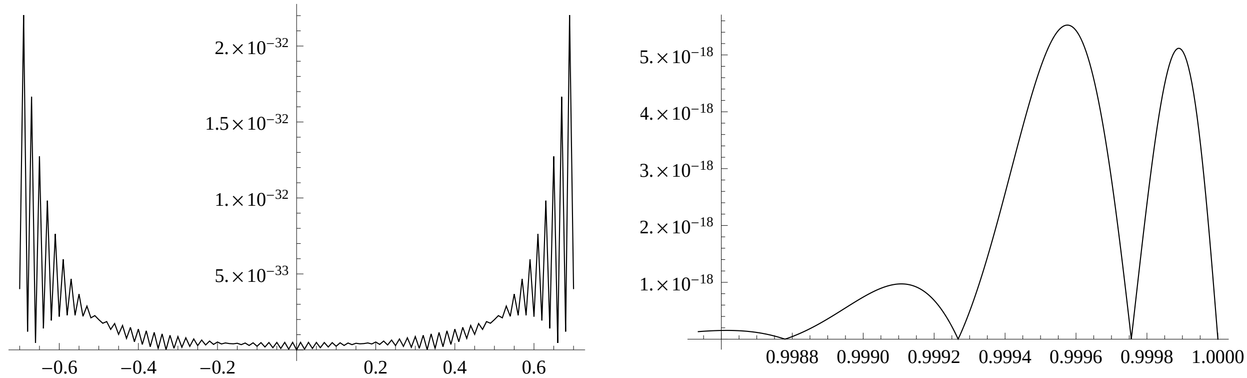

Figure 8: Graphs of $|r^p_{2048,q}(f;x)|$ on the interval $[-0.7, 0.7]$ (left) and at the point $x = 1$ (right) for function (3) when parameters τ_k are determined from (14) with $p = 2$ and $q = 4$.

Appendix

Here we present some lemmas concerning the properties of the generalized finite differences.

Lemma 1. *[13] The following estimate holds for $p > 0$ and $m \geq 0$*

$$\begin{aligned}\delta_n^p(\theta, B_n(m)) &= \frac{(-1)^{n+p+1}}{2(i\pi n)^{m+1} n^{2p} m!} \sum_{s=0}^{p} (-1)^s \frac{\gamma_s(\tau)}{N^s n^{-s}} \sum_{k=0}^{p} \frac{\gamma_k(\tau)}{N^k n^{-k}} \\ &\times \ (2p - k - s + m)! + \frac{1}{N^{2p}} O(n^{-m-2}), \ |n| \geq N+1, \ N \to \infty.\end{aligned}$$

Lemma 2. *[13] The following estimate holds for $p > 0$ and $m \geq 0$*

$$\begin{aligned}\delta_n^p(\theta, \check{B}_n(m) &- B_n(m)) = \frac{(-1)^{n+p+1}}{2(i\pi N)^{m+1} N^{2p} m!} \sum_{s=0}^{p} (-1)^s \gamma_s(\tau) \sum_{k=0}^{p} \gamma_k(\tau) \\ &\times \ (2p - k - s + m)! \sum_{r \neq 0} \frac{(-1)^r}{\left(2r + \frac{n}{N}\right)^{2p-k-s+m+1}} \\ &+ \ O(N^{-2p-m-2}), \ N \to \infty, \ |n| \leq N.\end{aligned}$$

Lemma 3. *[13] Let m be even. Then the following estimate holds for $p, w, m \geq 0$ as $N \to \infty$, where θ_k are chosen as in (1)*

$$\begin{aligned}
\delta^w_{\pm N}(\delta^p_n(\theta, \check{B}_n(m))) &= \frac{(-1)^{N+p+w+1}}{2(i\pi N)^{m+1}N^{2w+2p}m!}\sum_{s=0}^{p}(-1)^s\gamma_s(\tau)\sum_{k=0}^{p}\gamma_k(\tau) \\
&\times (2w+2p-k-s+m)!\sum_{r=-\infty}^{\infty}\frac{(-1)^r}{(2r\pm 1)^{2w+2p-k-s+m+1}} \\
&+ O(N^{-2w-2p-m-2}).
\end{aligned}$$

Lemma 4. *[13] Let m be odd. Then the following estimate holds for $p, w \geq 0$ and $m \geq 1$ as $N \to \infty$, where θ_k are chosen as in (1)*

$$\begin{aligned}
\delta^w_{\pm N}(\delta^p_n(\theta, \check{B}_n(m))) &= \frac{(-1)^{N+p+w}}{2(i\pi N)^{m+1}N^{2w+2p+1}m!}\sum_{s=0}^{p}(-1)^s\gamma_s(\tau)\sum_{k=0}^{p}\gamma_k(\tau) \\
&\times (2w+2p-k-s+m+1)!\sum_{r=-\infty}^{\infty}\frac{(-1)^r r}{(2r\pm 1)^{2w+2p-k-s+m+2}} \\
&+ O(N^{-2w-2p-m-3}).
\end{aligned}$$

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