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On Some Analytic Operator Functions in the Theory of Hermitian Operators

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Abstract

A densely defined Hermitian operator A_0 with equal defect numbers is considered. Presentable by means of resolvents of a certain maximal dissipative or accumulative extensions of A_0, bounded linear operators acting from some defect subspace $\mathfrak{N}_\gamma$ to arbitrary other $\mathfrak{N}_\lambda$ are investigated. With their aid are discussed characteristic and Weyl functions. A family of Weyl functions is described, associated with a given self-adjoint extension of A_0. The specific property of Weyl function's factors enabled to obtain a modified formulas of von Neumann. In terms of characteristic and Weyl functions of suitably chosen extensions the resolvent of Weyl function is presented explicitly.

Key Words: Hermitian operator, maximal extension, resolvent, characteristic function, Weyl function, von Neumann's formulas
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1. Introduction

The present paper concerns with maximal extensions of a closed, densely defined Hermitian operator A_0 with equal defect numbers. Characteristic function (Θ-function) of a maximal dissipative or accumulative extension, and Weyl function (M- function) of a self-adjoint extension are treated with the help of operator-valued function $\Theta(\gamma,\lambda)=\mathcal{P}_\gamma|\mathfrak{N}_\lambda$. Projection-valued function $\mathcal{P}_\gamma$ is given by direct-sum decompositions

$$\mathcal{D}\,(A_0^*) = \mathcal{D}(A_0) \dotplus \mathfrak{N}_\gamma \dotplus \mathfrak{N}_{\bar\gamma}, \qquad Im\,\gamma \neq 0.$$

Explicit presentation of both $\mathcal{P}_\gamma$ and $\Theta(\gamma,\lambda)$ in terms of resolvents of dissipative or accumulative extension $A_\gamma = A_0^*|Ker\mathcal{P}_{\bar\gamma}$ allowed to identify various definitions of Θ-function of

A_γ. The features of $\Theta(\gamma,\lambda)$ are suffice to present the main properties of M-function without employing the concept of space of abstract boundary values.

The content of this paper is as follows.

In Sec. 2 we consider extensions A_γ and their resolvents. Introducing $\Theta(\gamma,\lambda)$ operators and examining their properties we arrive at (2×2)-matrix function $W(\gamma,\lambda)$ with operator entries, which provides the Kreĭn-Shmulyan inter-spherical linear fractional transformation $\Phi_{W(\gamma,\lambda)}(\cdot)$ [6].

In Sec. 3 we briefly review and unify some basic material on Θ-functions. It is shown that Θ-function of A_γ in sense of Nagy-Foias [10], [4] coincides with that introduced by A. Straus [12, 13]. The same operator function appears when we apply to A_γ, $A_{\bar\gamma}$ operators the variant of definition of matrix Θ-function, developed by A. Kuzhel [7]. Θ-function of arbitrary maximal dissipative (accumulative) extension is reproduced in accordance with that, obtained by Kuzhel's approach in [8] for canonical differential operators.

In Sec. 4 we consider a pair $A_{\pm V}$ of self-adjoint extensions, defined by von Neumann's formulas with the help of isometries $\pm V(\gamma_0)\in[\mathfrak{N}_{\bar\gamma_0},\mathfrak{N}_{\gamma_0}]$, $Im\,\gamma_0>0$. It is shown that isometries $V_\pm(\gamma)=\Phi^*_{W(\gamma,\gamma_0)}\,(\pm V^*(\gamma_0))\in[\mathfrak{N}_{\bar\gamma},\mathfrak{N}_\gamma]$, $Im\,\gamma>0$ provide the same pair. The factors of $V_\pm(\gamma)$ build the Weyl function of A_V (A_{-V}) in definition of V. Derkach, M. Malamud [1], and we discuss the set of Weyl functions, corresponding to A_V. An analog of von Neumann formulas is derived, applied to the decomposition

$$\mathcal{D}\,(A_0^*)=\mathcal{D}(A_0)\dot{+}\mathfrak{N}_\gamma\dot{+}\mathfrak{N}_\zeta,\qquad Im\,\gamma\,Im\,\zeta<0.$$

In that the part of isometry is played by bounded invertible operator $M(\gamma,\zeta)\in[\mathfrak{N}_\zeta,\mathfrak{N}_\gamma]$ with the properties of M-function's corresponding factor. We present also one more proof of Kreĭn's resolvent formula (Kreĭn-Saakyan formula), which differs from those of [11, 1, 3], and is applicable to canonical differential operators [9]. Lastly we compute the resolvent of A_V operator's M-function, extending the corresponding formula of [9] from real axis onto the resolvent set.

2. Properties of $\Theta(\gamma,\lambda)$ operators

2.1.

Let $\mathfrak{H}$ be a Hilbert space with an inner product $\langle\cdot,\cdot\rangle$, and let A_0 be a closed Hermitian operator in $\mathfrak{H}$ with the domain $\mathcal{D}(A_0)$ dense in $\mathfrak{H}$. The complex plane C we consider with the standard subdivision $C=C^-\cup R\cup C^+$. A defect subspace of A_0 denote by $\mathfrak{N}_\gamma:=Ker\,(A_0^*-\gamma I)$, $\gamma\in C^-\cup C^+$ and assume that $\dim\mathfrak{N}_\gamma=\dim\mathfrak{N}_\zeta\leqslant\infty$, $Im\,\gamma\cdot Im\,\zeta<0$.

The direct-sum decomposition

$$\mathcal{D}\,(A_0^*)=\mathcal{D}(A_0)\dot{+}\mathfrak{N}_\gamma\dot{+}\mathfrak{N}_{\bar\gamma}\tag{2.1}$$

associates an oblique projection $\mathcal{P}_\gamma : \mathcal{D}(A_0^*) \to \mathfrak{N}_\gamma$ onto $\mathfrak{N}_\gamma$ along $\mathcal{D}(A_0) \dot{+} \mathfrak{N}_{\bar{\gamma}}$ to each γ, defining projection-valued function $\mathcal{P}_\gamma$ on $C^- \cup C^+$.

Denote $f_0 \in \mathcal{D}(A_0)$, $f_\gamma \in \mathfrak{N}_\gamma$ and introduce the operators

$$A_\gamma = A_0^* | \mathcal{D}(A_\gamma), \qquad \mathcal{D}(A_\gamma) = Ker \mathcal{P}_{\bar{\gamma}} = \mathcal{D}(A_0) \dot{+} \mathfrak{N}_\gamma,$$

that is

$$A_\gamma f = A_0 f_0 + \gamma f_\gamma, \qquad \text{if } f = f_0 + f_\gamma \in \mathcal{D}(A_\gamma). \tag{2.2}$$

Clearly, A_γ are extensions of A_0. If $Im\,\gamma \cdot Im\,\lambda > 0$, then operators $R\left(A_\gamma, \bar{\lambda}\right) := \left(A_\gamma - \bar{\lambda} I\right)^{-1}$ exist, $\mathcal{D}\left(\left(A_\gamma - \bar{\lambda} I\right)^{-1}\right) = Ran\left(A_\gamma - \bar{\lambda} I\right) = \mathfrak{H}$, hence A_γ are maximal extensions of A_0 (see [2], Sec 12.4, [5] Sec 3.1.2). Relations (2.2) yield $\langle A_\gamma f, g\rangle = \langle f, A_{\bar{\gamma}} g\rangle$ whenever $f \in \mathcal{D}(A_\gamma)$, $g \in \mathcal{D}(A_{\bar{\gamma}})$, therefore a maximality of $A_{\bar{\gamma}}$ implies $A_\gamma^* = A_{\bar{\gamma}}$.

Hermiteness of A_0 yields

$$Im\langle A_\gamma f, f\rangle = Im\,\gamma \langle f_\gamma, f_\gamma\rangle = Im\,\gamma \langle \mathcal{P}_\gamma f, \mathcal{P}_\gamma f\rangle, \qquad f \in \mathcal{D}(A_\gamma). \tag{2.3}$$

If $\gamma \in C^+$, formula (2.3) specifies that A_γ and $A_{\bar{\gamma}} = A_\gamma^*$ are maximal dissipative and accumulative extensions of Hermitian operator A_0. Their resolvent sets $\rho(A_\gamma)$ and $\rho(A_{\bar{\gamma}})$ comprise C^- and C^+ respectively.

From (2.2) we see that $Ran(A_\gamma - \gamma I) = Ran(A_0 - \gamma I)$, $Ker(A_\gamma - \gamma I) = \mathfrak{N}_\gamma$, hence the orthogonal decomposition

$$\mathfrak{H} = Ran(A_0 - \gamma I) \oplus Ker\left(A_0^* - \bar{\gamma} I\right) \tag{2.4}$$

can be presented also in the form

$$\mathfrak{H} = Ran(A_\gamma - \gamma I) \oplus Ker\left(A_{\bar{\gamma}} - \bar{\gamma} I\right). \tag{2.5}$$

Let P_γ be the orthogonal projection in $\mathfrak{H}$ onto $\mathfrak{N}_\gamma$. Projections $\mathcal{P}_\gamma$ and P_γ can be presented explicitly by means of A_γ.

Proposition 2.1 *Projections $\mathcal{P}_\gamma$ and P_γ are related by*

$$\mathcal{P}_\gamma = \frac{1}{\gamma - \bar{\gamma}} P_\gamma \left(A_0^* - \bar{\gamma} I\right). \tag{2.6}$$

Proof. For any $f = f_0 + f_\gamma + f_{\bar{\gamma}} \in \mathcal{D}(A_0^*)$ we have

$$\left(A_0^* - \bar{\gamma} I\right) f = (A_0 - \bar{\gamma} I) f_0 + (\gamma - \bar{\gamma}) f_\gamma,$$

hence from (2.4) we obtain $P_\gamma \left(A_0^* - \bar{\gamma} I\right) f = (\gamma - \bar{\gamma}) \mathcal{P}_\gamma f$, with the result.

Proposition 2.2 *The projection $\mathcal{P}_\gamma$ is presentable in the form*

$$\mathcal{P}_\gamma = I - \left(A_{\bar{\gamma}} - \gamma I\right)^{-1} \left(A_0^* - \gamma I\right). \tag{2.7}$$

Proof. Consider the operator $Q_\gamma = (A_{\bar\gamma} - \gamma I)^{-1}(A_0^* - \gamma I)$, $\mathcal{D}(Q_\gamma) = \mathcal{D}(A_0^*)$. Clearly $Ran\, Q_\gamma = Ran\,(A_{\bar\gamma} - \gamma I)^{-1} = \mathcal{D}(A_{\bar\gamma})$, and $Q_\gamma \mathfrak{N}_\gamma = 0$. From $(A_0^* - \gamma I)(A_{\bar\gamma} - \gamma I)^{-1} f = f$ for arbitrary $f \in \mathcal{D}(A_{\bar\gamma} - \gamma I)^{-1} = \mathfrak{H}$ it follows that $Q_\gamma^2 = Q_\gamma$. Thus Q_γ is an oblique projection in $\mathcal{D}(A_0^*)$ onto $\mathcal{D}(A_{\bar\gamma})$ along $\mathfrak{N}_\gamma$, so $I - Q_\gamma = \mathcal{P}_\gamma$, and the proof is complete.

To obtain the presentation of P_γ first we take note that $I - P_\gamma$ is an orthogonal projection in $\mathfrak{H} = Ran\,(A_{\bar\gamma} - \bar\gamma I) \oplus \mathfrak{N}_\gamma$ onto $Ran\,(A_{\bar\gamma} - \bar\gamma I) = Ran\,(A_0 - \bar\gamma I)$. Now consider the Cayley transform

$$T_\gamma = (A_\gamma - \gamma I)(A_\gamma - \bar\gamma I)^{-1} = I - (\gamma - \bar\gamma)(A_\gamma - \bar\gamma I)^{-1} \in [\mathfrak{H}], \tag{2.8}$$

which is a contraction $\|T_\gamma\| \leqslant 1$ (see [5] Sec 3.1.2), and $T_\gamma^* = I - (\bar\gamma - \gamma)(A_{\bar\gamma} - \gamma I)^{-1} = (A_{\bar\gamma} - \bar\gamma I)(A_{\bar\gamma} - \gamma I)^{-1} = T_{\bar\gamma}$.

Proposition 2.3 *The operator T_γ is a partial isometry with initial subspace $Ran\,(A_{\bar\gamma} - \bar\gamma I)$ and final subspace $Ran\,(A_\gamma - \gamma I)$.*

Proof. We have to prove that $Ker\, T_\gamma = \mathfrak{N}_\gamma$ and T_γ is isometric on $Ran\,(A_{\bar\gamma} - \bar\gamma I) = Ran\,(A_0 - \bar\gamma I)$. Let $f_\gamma \in \mathfrak{N}_\gamma$. Then $(A_\gamma - \bar\gamma I)^{-1} f_\gamma = g_0 + g_\gamma \in \mathcal{D}(A_\gamma)$, so

$$(A_\gamma - \bar\gamma I)(g_0 + g_\gamma) = (A_0 - \bar\gamma I)\, g_0 + (\gamma - \bar\gamma)\, g_\gamma = f_\gamma,$$

hence

$$g_\gamma = \frac{1}{\gamma - \bar\gamma} f_\gamma\,; \qquad (A_0 - \bar\gamma I)\, g_0 = 0, \quad g_0 = 0.$$

Thus $T_\gamma f_\gamma = (A_\gamma - \gamma I)\, g_\gamma = 0$ for any $f_\gamma \in \mathfrak{N}_\gamma$, and $\mathfrak{N}_\gamma \subset Ker T_\gamma$. On the other hand, for arbitrary $f_0 \in \mathcal{D}(A_0)$ one has $T_\gamma (A_0 - \bar\gamma I) f_0 = (A_0 - \gamma I) f_0$, hence $T_\gamma^* (A_0 - \gamma I) f_0 = (A_0 - \bar\gamma I)\, g_0$. Thus $T_\gamma^* T_\gamma f = f$ for any $f = (A_0 - \bar\gamma I) f_0 \in Ran\,(A_0 - \bar\gamma I)$, which completes the proof.

The self-adjoint operator $T_\gamma^* T_\gamma$, is an orthogonal projection, since evidently $\left(T_\gamma^* T_\gamma\right)^2 = T_\gamma^* T_\gamma$. Thus we have

$$T_\gamma^* T_\gamma = I - P_\gamma, \qquad P_\gamma = I - T_\gamma^* T_\gamma. \tag{2.9}$$

2.2.

Let γ, λ be non-real numbers. Denote $\Theta(\gamma, \lambda) := \mathcal{P}_\gamma | \mathfrak{N}_\lambda$ and observe that from (2.6) it follows that

$$\Theta(\gamma, \lambda) = \frac{\lambda - \bar\gamma}{\gamma - \bar\gamma} P_\gamma | \mathfrak{N}_\lambda\,. \tag{2.10}$$

Proposition 2.4 $\Theta(\gamma, \lambda)$ *operators are presentable as*

$$\Theta(\gamma, \lambda) = (A_{\bar\gamma} - \lambda I)(A_{\bar\gamma} - \gamma I)^{-1} | \mathfrak{N}_\lambda\,. \tag{2.11}$$

Proof. Formula (2.7) applied to $f_\lambda \in \mathfrak{N}_\lambda$ immediately leads to the result

$$\mathcal{P}_\gamma f_\lambda = f_\lambda - (\lambda - \gamma)\left(A_{\bar\gamma} - \gamma I\right)^{-1} f_\lambda = \left[(A_{\bar\gamma} - \gamma I) - (\lambda - \gamma) I\right]\left(A_{\bar\gamma} - \gamma I\right)^{-1} f_\lambda.$$

If $Im\,\gamma \cdot Im\,\lambda > 0$ it now follows that $\Theta(\gamma, \lambda)$ is bounded invertible and

$$\Theta^{-1}(\gamma, \lambda) = \left(A_{\bar\gamma} - \gamma I\right)\left(A_{\bar\gamma} - \lambda I\right)^{-1} |\mathfrak{N}_\gamma. \tag{2.12}$$

Formula (2.11) also implies that, if $Im\,\gamma \cdot Im\,\lambda < 0$, then the operator $\Theta(\gamma, \lambda)$ is not invertible if and only if λ is an eigenvalue of $A_{\bar\gamma}$, that is, if λ is in a point spectrum $\sigma_p\left(A_{\bar\gamma}\right)$ of $A_{\bar\gamma}$.

Applying decomposition (2.1) to arbitrary $f_\lambda \in \mathfrak{N}_\lambda$ we have $f_\lambda = f_0 + f_\gamma + f_{\bar\gamma}$, where $f_\gamma = \mathcal{P}_\gamma f_\lambda$, $f_{\bar\gamma} = \mathcal{P}_{\bar\gamma} f_\lambda$, so it holds the identity

$$f_\lambda = f_0 + \Theta(\gamma, \lambda) f_\lambda + \Theta(\bar\gamma, \lambda) f_\lambda. \tag{2.13}$$

In the case $Im\,\gamma \cdot Im\,\lambda > 0$ the formula above can be written in the form

$$f_\lambda = f_0 + f_\gamma + \Theta_\gamma(\lambda) f_\gamma, \quad \Theta_\gamma(\lambda) := \Theta\left(\bar\gamma, \lambda\right)\Theta^{-1}(\gamma, \lambda) \in \left[\mathfrak{N}_\gamma, \mathfrak{N}_{\bar\gamma}\right]. \tag{2.14}$$

Now we collect the main properties of $\Theta(\gamma, \lambda)$ necessary in what follows.

Proposition 2.5 *If γ, λ, ζ are arbitrary non-real numbers, then*

$$\Theta(\gamma, \zeta) = \Theta(\gamma, \lambda)\Theta(\lambda, \zeta) + \Theta\left(\gamma, \bar\lambda\right)\Theta\left(\bar\lambda, \zeta\right). \tag{2.15}$$

Proof. Without loss of generality we assume that $Im\,\gamma \cdot Im\,\lambda > 0$, since λ and $\bar\lambda$ appear symmetrically in the formula to be proved. In the presence of (2.13) we have

$$f_\zeta = g_0 + g_\gamma + g_{\bar\gamma} = h_0 + h_\lambda + h_{\bar\lambda},$$

where $g_\gamma = \Theta(\gamma, \zeta) f_\zeta$, $g_{\bar\gamma} = \Theta\left(\bar\gamma, \zeta\right) f_\zeta$, $h_\lambda = \Theta(\lambda, \zeta) f_\zeta$, $h_{\bar\lambda} = \Theta\left(\bar\lambda, \zeta\right) f_\zeta$. In its turn

$$h_\lambda = u_0 + u_\gamma + u_{\bar\gamma}\,, \qquad h_{\bar\lambda} = v_0 + v_\gamma + v_{\bar\gamma}\,,$$

therefore from (2.13) we get

$$g_\gamma = u_\gamma + v_\gamma = u_\gamma + \Theta\left(\gamma, \bar\lambda\right)\Theta^{-1}(\bar\gamma, \bar\lambda) v_{\bar\gamma}.$$

Since $u_\gamma = \Theta(\gamma, \lambda) h_\lambda = \Theta(\gamma, \lambda)\Theta(\lambda, \zeta) f_\zeta$, $v_{\bar\gamma} = \Theta(\bar\gamma, \bar\lambda) h_{\bar\lambda} = \Theta\left(\bar\gamma, \bar\lambda\right)\Theta(\bar\lambda, \zeta) f_\zeta$, the formula above leads to

$$\Theta(\gamma, \zeta) f_\zeta = g_\gamma = \Theta(\gamma, \lambda)\Theta(\lambda, \zeta) f_\zeta + \Theta\left(\gamma, \bar\lambda\right)\Theta^{-1}\left(\bar\gamma, \bar\lambda\right)\Theta\left(\bar\gamma, \bar\lambda\right)\Theta\left(\bar\lambda, \zeta\right) f_\zeta,$$

and (2.15) results.

In particular, setting in (2.15) first $\zeta = \gamma$, and then $\zeta = \bar\gamma$, we obtain

$$\begin{aligned} \Theta(\gamma, \lambda)\Theta(\lambda, \gamma) + \Theta(\gamma, \bar\lambda)\Theta(\bar\lambda, \gamma) &= \Theta(\gamma, \gamma) = I_\gamma \\ \Theta(\gamma, \lambda)\Theta(\lambda, \bar\gamma) + \Theta(\gamma, \bar\lambda)\Theta(\bar\lambda, \bar\gamma) &= \Theta(\gamma, \bar\gamma) = 0. \end{aligned} \tag{2.16}$$

The next property of $\Theta(\gamma, \lambda)$ is the following statement.

Proposition 2.6 *The operator adjoint to $\Theta(\gamma,\lambda)$ is presented by*

$$\Theta^*(\gamma,\lambda) = \frac{Im\,\lambda}{Im\,\gamma}\,\Theta(\lambda,\gamma). \tag{2.17}$$

Proof. First, let us recall that in the proof of Proposition 2.3 we have seen the relation $\left(A_\lambda - \bar{\lambda} I\right)^{-1} f_\lambda = \left(\lambda - \bar{\lambda}\right)^{-1} f_\lambda$, hence

$$\left(A_\lambda - \bar{\gamma} I\right)\left(A_\lambda - \bar{\lambda} I\right)^{-1} |\mathfrak{N}_\lambda = \frac{\lambda - \bar{\gamma}}{\lambda - \bar{\lambda}}\, I_\lambda.$$

Thus from (2.10) we have

$$\langle \Theta\left(\gamma,\lambda\right) f_\lambda, g_\gamma\rangle = \frac{\lambda-\bar{\lambda}}{\lambda-\bar{\gamma}}\langle \left(A_{\bar{\gamma}} - \lambda I\right)\left(A_{\bar{\gamma}} - \gamma I\right)^{-1}\left(A_\lambda - \bar{\gamma} I\right)\left(A_\lambda - \bar{\lambda} I\right)^{-1} f_\lambda, g_\gamma\rangle =$$

$$= \frac{\lambda-\bar{\lambda}}{\lambda-\bar{\gamma}}\langle \left(A_\lambda - \bar{\gamma} I\right)\left(A_\lambda - \bar{\lambda} I\right)^{-1} f_\lambda, \left(A_\gamma - \bar{\lambda} I\right)\left(A_\gamma - \bar{\gamma} I\right)^{-1} g_\gamma\rangle =$$

$$= \frac{\lambda-\bar{\lambda}}{\lambda-\bar{\gamma}}\,\frac{\bar{\gamma}-\lambda}{\bar{\gamma}-\gamma}\langle \left(A_\lambda - \bar{\gamma} I\right)\left(A_\lambda - \bar{\lambda} I\right)^{-1} f_\lambda, g_\gamma\rangle = \frac{Im\,\lambda}{Im\,\gamma}\langle f_\lambda, \left(A_{\bar{\lambda}} - \gamma I\right)\left(A_{\bar{\lambda}} - \lambda I\right)^{-1} g_\gamma\rangle,$$

and formula (2.17) is proved.

Now assume $Im\,\gamma \cdot Im\,\lambda > 0$ and consider the function $\Theta_\gamma(\lambda) = \Theta\left(\bar{\gamma},\lambda\right)\Theta^{-1}\left(\gamma,\lambda\right)$, appeared in (2.14). Clearly $\Theta_\gamma(\gamma) = 0$. Combining Proposition 2.5 and Proposition 2.6 we complete the list of necessary facts on $\Theta(\gamma,\lambda)$ operators.

Proposition 2.7 *The operator $\Theta_\gamma(\lambda)$ is a strict contraction $\|\Theta_\gamma(\lambda)\| < 1$, and it holds that $\Theta^*_\gamma(\lambda) = \Theta_{\bar{\gamma}}\left(\bar{\lambda}\right)$.*

Proof. Rewrite identities (2.17) in the form

$$\Theta(\lambda,\gamma)\Theta(\gamma,\lambda) + \Theta\left(\lambda,\bar{\gamma}\right)\Theta\left(\bar{\gamma},\lambda\right) = I_\lambda; \quad \Theta(\bar{\lambda},\bar{\gamma})\Theta(\bar{\gamma},\lambda) + \Theta\left(\bar{\lambda},\gamma\right)\Theta\left(\gamma,\lambda\right) = 0.$$

In view of (2.17) we obtain

$$\Theta^*(\gamma,\lambda)\Theta(\gamma,\lambda) - \Theta^*\left(\bar{\gamma},\lambda\right)\Theta\left(\bar{\gamma},\lambda\right) = \frac{Im\,\lambda}{Im\,\gamma}\, I_\lambda; \quad \Theta^*(\bar{\gamma},\bar{\lambda})\Theta(\bar{\gamma},\lambda) - \Theta^*\left(\gamma,\bar{\lambda}\right)\Theta\left(\gamma,\lambda\right) = 0.$$

The first identity above can be presented as

$$I_\gamma - \Theta^*_\gamma(\lambda)\Theta_\gamma(\lambda) = \frac{Im\,\lambda}{Im\,\gamma}\left[\Theta(\gamma,\lambda)\Theta^*(\gamma,\lambda)\right]^{-1} > 0,$$

and the second is

$$\Theta_\gamma(\lambda) = \Theta\left(\bar{\gamma},\lambda\right)\Theta^{-1}(\gamma,\lambda) = \Theta^{-*}(\bar{\gamma},\bar{\lambda})\Theta^*(\gamma,\bar{\lambda}) = \left[\Theta\left(\gamma,\bar{\lambda}\right)\Theta^{-1}\left(\bar{\gamma},\bar{\lambda}\right)\right]^* = \Theta^*_{\bar{\gamma}}\left(\bar{\lambda}\right).$$

The proof is complete.

2.3.

Identities (2.17) suggest the following construction. Let γ, λ be arbitrary in C^+. Introduce the Hilbert space $\mathcal{N}_{(\gamma)}$ of pairs $\mathbf{f}_{(\gamma)} = (f_\gamma, f_{\bar\gamma})$, $f_\gamma \in \mathfrak{N}_\gamma$, $f_{\bar\gamma} \in \mathfrak{N}_{\bar\gamma}$ with the inner product $\langle \mathbf{f}_{(\gamma)}, \mathbf{g}_{(\gamma)} \rangle = \langle f_\gamma, g_\gamma \rangle + \langle f_{\bar\gamma}, g_{\bar\gamma} \rangle$. Analogously is defined $\mathcal{N}_{(\lambda)}$. Consider the operator

$$W(\gamma,\lambda) = \begin{bmatrix} \Theta(\gamma,\lambda) & \Theta\left(\gamma,\bar\lambda\right) \\ \Theta\left(\bar\gamma,\lambda\right) & \Theta(\bar\gamma,\bar\lambda) \end{bmatrix} \in \left[\mathcal{N}_{(\lambda)}, \mathcal{N}_{(\gamma)}\right]. \tag{2.18}$$

Relations (2.16) imply that

$$W^{-1}(\gamma,\lambda) = \begin{bmatrix} \Theta(\lambda,\gamma) & \Theta\left(\lambda,\bar\gamma\right) \\ \Theta\left(\bar\lambda,\gamma\right) & \Theta(\bar\lambda,\bar\gamma) \end{bmatrix} = W(\lambda,\gamma).$$

If

$$\mathcal{J}_\gamma = \begin{bmatrix} I_\gamma & 0 \\ 0 & -I_\gamma \end{bmatrix} \in [\mathcal{N}_\gamma], \quad \mathcal{J}_\gamma^2 = I_{(\gamma)}, \quad \mathcal{J}_\gamma^* = \mathcal{J}_\gamma,$$

then from (2.17) it follows that

$$W^*(\gamma,\lambda) = \frac{Im\,\lambda}{Im\,\gamma} \begin{bmatrix} \Theta(\lambda,\gamma) & -\Theta\left(\lambda,\bar\gamma\right) \\ -\Theta\left(\bar\lambda,\gamma\right) & \Theta(\bar\lambda,\bar\gamma) \end{bmatrix} = \alpha \mathcal{J}_\lambda W(\lambda,\gamma)\mathcal{J}_\gamma, \quad \alpha = \frac{Im\,\lambda}{Im\,\gamma} > 0.$$

The last formula can be presented as

$$\tilde W^*(\gamma,\lambda)\mathcal{J}_\gamma \tilde W(\gamma,\lambda) = \mathcal{J}_\lambda, \quad \text{where} \quad \tilde W(\gamma,\lambda) = \alpha^{-\frac{1}{2}} W(\gamma,\lambda),$$

so $W(\gamma,\lambda)$ is collinear to $(\mathcal{J}_\lambda, \mathcal{J}_\gamma)$ – unitary operator $\tilde W(\gamma,\lambda)$.

With a $\mathcal{J}$-unitary operator W acting in Kreĭn space $\mathcal{H}$ associates the Kreĭn-Shmulyan linear fractional transformation $\Phi_W(\cdot)$, which has an inter-spherical property (see [6]).

The main statements of referred above remain valid also for the case under consideration. Namely, let $K_\lambda \in [\mathfrak{N}_\lambda, \mathfrak{N}_{\bar\lambda}]$. Denote

$$\Phi_{\bar\gamma}(K_\lambda) = \Theta\left(\bar\gamma,\lambda\right) + \Theta\left(\bar\gamma,\bar\lambda\right) K_\lambda, \qquad \Phi_\gamma(K_\lambda) = \Theta\left(\gamma,\lambda\right) + \Theta\left(\gamma,\bar\lambda\right) K_\lambda. \tag{2.19}$$

If $\Phi_\gamma(K_\lambda)$ is bounded invertible, the Kreĭn-Shmulyan transformation is

$$\Phi_{W(\gamma,\lambda)}(K_\lambda) = \Phi_{\bar\gamma}(K_\lambda)\Phi_\gamma^{-1}(K_\lambda). \tag{2.20}$$

Apparently $\Phi_{\tilde W(\gamma,\lambda)}(K_\lambda) = \Phi_{W(\gamma,\lambda)}(K_\lambda)$.

The inter-spherical property of $W(\gamma,\lambda)$ is characterized as follows.

Proposition 2.8 *Let $W(\gamma,\lambda)$ be given as in* (2.18)*. If $\|K_\lambda\| \leqslant 1$, then $K_\gamma = \Phi_{W(\gamma,\lambda)}(K_\lambda)$ is well defined, $\|K_\gamma\| \leqslant 1$, and K_λ, K_γ are isometries simultaneously.*

Proof. The second identity of (2.16) yields

$$\Theta^{-1}(\gamma,\lambda)\Theta(\gamma,\bar{\lambda}) = -\Theta(\lambda,\bar{\gamma})\Theta^{-1}\left(\bar{\lambda},\bar{\gamma}\right) = -\Theta_{\bar{\lambda}}\left(\bar{\gamma}\right),$$

hence

$$\Phi_\gamma\left(K_\lambda\right) = \Theta(\gamma,\lambda)\left[I_\lambda + \Theta^{-1}(\gamma,\lambda)\Theta(\gamma,I)K_\lambda\right] = \Theta(\gamma,\lambda)\left[I_\lambda - \Theta_{\bar{\lambda}}(\bar{\gamma})K_\lambda\right].$$

If $\|K_\lambda\| \leqslant 1$, then from Proposition 2.7 we have $\|\Theta_{\bar{\lambda}}(\bar{\gamma})K_\lambda\| < 1$, which proves bounded invertibility of $\Phi_\gamma(K_\lambda)$.

Now consider

$$I_\gamma - K_\gamma^* K_\gamma = \Phi_\gamma^{-*}(K_\lambda)\left[\Phi_\gamma^*(K_\lambda)\Phi_\gamma(K_\lambda) - \Phi_{\bar{\gamma}}^*(K_\lambda)\Phi_{\bar{\gamma}}(K_\lambda)\right]\Phi_\gamma^{-1}(K_\lambda).$$

Again making use (2.16), (2.17), by straightforward computations one can verify that

$$I_\gamma - K_\gamma^* K_\gamma = \frac{Im\,\lambda}{Im\,\gamma}\,\Phi_\gamma^{-*}(K_\lambda)\left[I_\lambda - K_\lambda^* K_\lambda\right]\Phi_\gamma^{-1}(K_\lambda),$$

which completes the proof.

3. Characteristic functions of maximal extensions

3.1.

Here we review and unify some fundamentals on characteristic functions, taken from [4, 7, 10, 12, 13].

First we turn to Θ-function in sense of Nagy-Foias and recall a few basic notions and facts from [10] (Sec 1.3, 6.1).

Let T be a contraction in a Hilbert space $\mathcal{H}$. Defect operators of T are $D_T = (I - T^*T)^{\frac{1}{2}}$, $D_{T^*} = (I - TT^*)^{\frac{1}{2}}$; defect subspaces of T are $\mathfrak{D}_T = \overline{D_T\mathcal{H}}$, $\mathfrak{D}_{T^*} = \overline{D_{T^*}\mathcal{H}}$. The following relations are valid:

$$TD_T = D_{T^*}T, \qquad T^*D_{T^*} = D_TT^*. \tag{3.1}$$

The Nagy-Foias characteristic function of T is an operator-valued function defined by

$$\Theta_T(\omega) = \left[-T + \sum_{n=1}^{\infty}\omega^n D_{T^*}T^{*\,n-1}D_T\right]|\mathfrak{D}_T, \quad |\omega| < 1, \tag{3.2}$$

hence is analytical in the unit disk $|\omega| < 1$. A left-hand multiplication of both sides of (3.2) by D_{T^*} and use of (3.1) leads to the equivalent definition

$$D_{T^*}\Theta_T(\omega) = (\omega I - T)\left(I - \omega T^*\right)^{-1}|\mathfrak{D}_T.$$

Values of $\Theta_T(\omega)$ are in $[\mathfrak{D}_T, \mathfrak{D}_{T^*}]$ and $\|\Theta_T(0)h\| < \|h\|$, $h \in \mathfrak{D}_T$.

Θ-function of dissipative operator $B =: (I+T)(I-T)^{-1}$ is defined by the relation $\Theta_B(\lambda) := \Theta_T\left(\dfrac{\lambda-i}{\lambda+i}\right)$, $\lambda \in C^+$ (see [4] Sec 28.12).

From now on we attach $\gamma, \lambda \in C^+$, $\zeta \in C^-$.

Let A_γ be the maximal dissipative operator given in (2.2).

Proposition 3.1 *If* $T_\gamma = (A_\gamma - \gamma I)(A_\gamma - \bar{\gamma} I)^{-1}$, *then*

$$\Theta_{T_\gamma}(\omega) = \omega P_{\bar{\gamma}}\left(I - \omega T_\gamma^*\right)^{-1} |\mathfrak{N}_\gamma = (\omega I - T_\gamma)\left(I - \omega T_\gamma^*\right)^{-1} |\mathfrak{N}_\gamma, \tag{3.3}$$

and $\Theta_{T_\gamma}(\omega)$ *admits the estimate*

$$\left\|\Theta_{T_\gamma}(\omega)\right\| \leqslant |\omega|, \qquad |\omega| < 1. \tag{3.4}$$

Proof. Referring back to Proposition 2.3 we see, that defect operators and defect subspaces of partially isometric operator T_γ are orthogonal projections P_γ, $P_{\bar{\gamma}}$ on subspaces $\mathfrak{N}_\gamma$, $\mathfrak{N}_{\bar{\gamma}}$ respectively. Moreover, formula (3.1) now takes the form

$$T_\gamma P_\gamma = P_{\bar{\gamma}} T_\gamma = 0, \qquad T_\gamma^* P_{\bar{\gamma}} = P_\gamma T_\gamma^* = 0,$$

hence definition (3.2) for the operator T_γ appears as

$$\Theta_{T_\gamma}(\omega) = \omega P_{\bar{\gamma}} \sum_{n=0}^{\infty} \omega^n T_\gamma^{*\,n} P_\gamma = \omega P_{\bar{\gamma}}\left(1 - \omega T_\gamma^*\right)^{-1} P_\gamma. \tag{3.5}$$

Since $P_{\bar{\gamma}}^2 = P_{\bar{\gamma}} = I - T_\gamma T_\gamma^*$, the last formula can be transformed to

$$\begin{gathered}\Theta_{T_\gamma}(\omega) = \omega P_{\bar{\gamma}}\left(I - T_\gamma T_\gamma^*\right) \sum_{n=1}^{\infty} \omega^n T_\gamma^{*\,n} P_\gamma = P_{\bar{\gamma}}\left[\omega\left(I - \omega T_\gamma^*\right)^{-1} - T_\gamma \sum_{n=1}^{\infty} \omega^n T_\gamma^{*\,n}\right] P_\gamma = \\ = -P_{\bar{\gamma}}\left\{\omega\left(I - \omega T_\gamma^*\right)^{-1} - T_\gamma\left[\left(I - \omega T_\gamma^*\right)^{-1} - I\right]\right\} P_\gamma = P_{\bar{\gamma}}(\omega I - T_\gamma)\left(I - \omega T_\gamma^*\right)^{-1} P_\gamma,\end{gathered} \tag{3.6}$$

again by virtue of $T_\gamma P_\gamma = 0$.

Analogously, because of $P_\gamma^2 = P_\gamma = I - T_\gamma^* T_\gamma$ and $T_\gamma^* P_{\bar{\gamma}} = 0$, we obtain

$$\Theta_{T_\gamma}(\omega) = \omega P_{\bar{\gamma}} \sum_{n=0}^{\infty} \omega^n T_\gamma^{*\,n}\left(I - T_\gamma^* T_\gamma\right) P_\gamma = P_{\bar{\gamma}}\left(I - \omega T_\gamma^*\right)^{-1} (\omega I - T_\gamma) P_\gamma. \tag{3.7}$$

From (3.5) we have $\Theta_{T_\gamma}(0) = 0$, hence $\Theta_{T_\gamma}(\lambda)$, analytical in $|\omega| < 1$, meets conditions of Shwartz lemma, which leads to (3.4). The proof is complete.

Proposition 3.2 *Characteristic function of maximal dissipative operator* A_γ *is analytical in* C^+ *operator function, given by the formula*

$$\begin{gathered}\Theta_{A_\gamma}(\lambda) = -(A_\gamma - \lambda I)(A_\gamma - \bar{\gamma} I)^{-1}(A_{\bar{\gamma}} - \gamma I)(A_{\bar{\gamma}} - \lambda I)^{-1} |\mathfrak{N}_\gamma = \\ = -(A_{\bar{\gamma}} - \gamma I)(A_{\bar{\gamma}} - \lambda I)^{-1}(A_\gamma - \lambda I)(A_\gamma - \bar{\gamma} I)^{-1} |\mathfrak{N}_\gamma,\end{gathered} \tag{3.8}$$

and satisfying the estimate

$$\left\|\Theta_{A_\gamma}(\lambda)\right\| \leqslant \left|\frac{\lambda-\gamma}{\lambda-\bar{\gamma}}\right|, \qquad \lambda \in C^+. \tag{3.9}$$

Proof. The first factor of (3.3) can be transformed as follows:

$$\omega I - T_\gamma = \omega I - (A_\gamma - \gamma I)(A_\gamma - \bar{\gamma} I)^{-1} = [\omega (A_\gamma - \bar{\gamma} I) - (A_\gamma - \gamma I)](A_\gamma - \bar{\gamma} I)^{-1} =$$

$$= [(\omega-1)A_\gamma + (\gamma - \omega\bar{\gamma}) I](A_\gamma - \bar{\gamma} I) = (1-\omega)\left(A_\gamma - \frac{\gamma - \omega\bar{\gamma}}{1-\omega} I\right)(A_\gamma - \bar{\gamma} I)^{-1}.$$

Similarly, the second factor is

$$I - \omega T_\gamma^* = [(1-\omega)A_{\bar{\gamma}} - (\gamma - \omega\bar{\gamma})](A_{\bar{\gamma}} - \gamma I)^{-1} = (1-\omega)\left(A_{\bar{\gamma}} - \frac{\gamma - \omega\bar{\gamma}}{1-\omega} I\right)(A_{\bar{\gamma}} - \gamma I)^{-1}.$$

Linear fractional transformation $\lambda = \dfrac{\gamma - \omega\bar{\gamma}}{1-\omega}$ maps the unit disk $|\omega| < 1$ onto the half-plane C^+, hence formula (3.3) can be rewritten as

$$\Theta_{T_\gamma}\left(\frac{\lambda-\gamma}{\lambda-\bar{\gamma}}\right) = \Theta_{A_\gamma}(\lambda) = -(A_\gamma - \lambda I)(A_\gamma - \bar{\gamma} I)^{-1}(A_{\bar{\gamma}} - \gamma I)(A_{\bar{\gamma}} - \lambda I)^{-1} P_\gamma.$$

The next equality of (3.8) follows analogously from (3.7).

Inequality (3.4) now takes the form (3.9), completing the proof.

A comparison of (3.8) with (2.11), (2.12) brings the relation

$$\Theta_{A_\gamma}(\lambda) = -\Theta(\bar{\gamma}, \lambda)\,\Theta^{-1}(\gamma, \lambda) = -\Theta_\gamma(\lambda). \tag{3.10}$$

The case of maximal accumulative extension $A_{\bar{\gamma}} = A_\gamma^*$ is treated analogously, and corresponding formulas are

$$\Theta_{A_{\bar{\gamma}}}(\zeta) = -\Theta(\gamma, \zeta)\Theta^{-1}\left(\gamma, \bar{\zeta}\right) = -\Theta_{\bar{\gamma}}(\zeta); \quad \left\|\Theta_{A_{\bar{\gamma}}}(\zeta)\right\| \leqslant \left|\frac{\zeta-\bar{\gamma}}{\zeta-\gamma}\right|.$$

From Proposition 2.7 it follows that $\Theta_{A_\gamma}^*(\lambda) = \Theta_{A_{\bar{\gamma}}}\left(\bar{\lambda}\right)$.

Another approach to Θ-functions is due to A. V. Straus [12, 13]. In [13] was introduced characteristic function of Hermitian operator A_0 as a contractive operator function

$$\tilde{\Theta}_\gamma(\lambda) = (A_\lambda - \gamma I)(A_\lambda - \bar{\gamma} I)^{-1} | \mathfrak{N}_\gamma \in [\mathfrak{N}_\gamma, \mathfrak{N}_{\bar{\gamma}}],$$

where $\gamma \in C^+$ is fixed, λ varies on C^+. The operator $\tilde{\Theta}_\gamma(\lambda)$ possesses the property:

$$\text{if } \ f_\lambda = f_0 + f_\gamma - \tilde{\Theta}_\gamma(\lambda) f_\gamma \ \text{ varies on } \ \mathfrak{N}_\lambda, \ \text{ then } \ f_\gamma \ \text{ ranges over the whole } \ \mathfrak{N}_\gamma. \tag{3.11}$$

We wish to prove that the same property holds for the case of decomposition (2.14). In view of (2.13) it is suffice to show that for arbitrary f_γ there exist f_λ such that $f_\gamma = \Theta(\gamma, \lambda) f_\lambda$.

Choose $g_\lambda = \Theta(\lambda,\gamma) f_\gamma$, so $f_\gamma = \Theta^{-1}(\lambda,\gamma) g_\lambda$. Rearranging γ and λ in (2.16) it is readily seen that $\Theta(\gamma,\lambda) = \Theta^{-1}(\lambda,\gamma)\left[I_\lambda - \Theta(\lambda,\bar\gamma)\Theta(\bar\gamma,\lambda)\right]$, hence $f_\lambda = \left[I_\lambda - \Theta(\lambda,\bar\gamma)\Theta(\bar\gamma,\lambda)\right]^{-1} g_\lambda$ is desirable vector.

Comparing (3.11) with (2.14) and noting uniqueness of decomposition $f_\lambda = f_0 + f_\gamma + f_{\bar\gamma}$ we conclude that

$$\tilde\Theta_\gamma(\lambda) = -\Theta_\gamma(\lambda) = \Theta_{A_\gamma}(\lambda). \tag{3.12}$$

In [12] was presented Θ-function of A_γ by the following manner.

Let $S_\gamma(\lambda) = \left(A_{\bar\gamma} - \lambda I\right)^{-1}\left(A_\gamma - \lambda I\right)$, and let $\tilde{\mathcal{P}}_\gamma$, $\tilde{\mathcal{P}}_{\bar\gamma}$ be oblique projections in $\mathcal{D}(A_\gamma)$, $\mathcal{D}\left(A_{\bar\gamma}\right)$ onto $\mathfrak{N}_\gamma$, $\mathfrak{N}_{\bar\gamma}$ respectively. Then Θ-function $\tilde\Theta_{A_\gamma}(\lambda)$ is defined by the formula

$$\tilde\Theta_{A_\gamma}(\lambda)\tilde{\mathcal{P}}_\gamma f = \tilde{\mathcal{P}}_{\bar\gamma} S_\gamma(\lambda) f, \qquad f \in \mathcal{D}\left(A_\gamma\right). \tag{3.13}$$

In [13] was noted that $\tilde\Theta_{A_\gamma}(\lambda) = \tilde\Theta_\gamma(\lambda)$, but no proof presented. To show it let us observe that

$$\tilde{\mathcal{P}}_\gamma f = \frac{1}{\gamma - \bar\gamma}\, P_\gamma \left(A_\gamma - \bar\gamma I\right) f.$$

Indeed, it is clear that $\tilde{\mathcal{P}}_\gamma^2 = \tilde{\mathcal{P}}_\gamma$. Applying (2.6) to $f \in \mathcal{D}(A_\gamma)$ we get

$$\mathcal{P}_\gamma f = \frac{1}{\gamma - \bar\gamma}\, P_\gamma\left(A_0^* - \bar\gamma I\right) f = \frac{1}{\gamma - \bar\gamma}\, P_\gamma\left(A_\gamma - \bar\gamma I\right) f = \tilde{\mathcal{P}}_\gamma f.$$

Similarly $\tilde{\mathcal{P}}_{\bar\gamma} g = -\dfrac{1}{\gamma - \bar\gamma}\left(A_{\bar\gamma} - \gamma I\right) g$, $g \in \mathcal{D}\left(A_{\bar\gamma}\right)$, hence (3.13) takes the form

$$\tilde\Theta_{A_\gamma}(\lambda) = -\left(A_{\bar\gamma} - \gamma I\right) S_\gamma(\lambda) \left(A_\gamma - \bar\gamma I\right)^{-1} |\mathfrak{N}_\gamma,$$

which is the right hand of (3.8).

Lastly, we discuss Θ-functions of A_γ and $A_{\bar\gamma}$ in definition, which is due to A. Kuzhel [7]. We shall extend to the case under consideration the method, introduced in [7] (Sec 2.1, 2.3) for the case of finite deficiency index (n,n).

Proposition 3.3 *Let $f_\lambda \in \mathfrak{N}_\lambda$ and $f_\zeta \in \mathfrak{N}_\zeta$ be arbitrary, and denote*

$$S\left(\bar\lambda,\bar\zeta\right) = \Theta\left(\bar\lambda,\gamma\right)\Theta^{-1}\left(\bar\zeta,\gamma\right), \quad S\left(\bar\zeta,\bar\lambda\right) = \Theta\left(\bar\zeta,\bar\gamma\right)\Theta^{-1}\left(\bar\lambda,\bar\gamma\right), \qquad \gamma \in C^+. \tag{3.14}$$

Then there exist $f_{\bar\zeta} \in \mathfrak{N}_{\bar\zeta}$ and $f_{\bar\lambda} \in \mathfrak{N}_{\bar\lambda}$ such that

$$f_\lambda + S\left(\bar\lambda,\bar\zeta\right) f_{\bar\zeta} \in \mathcal{D}\left(A_\gamma\right); \qquad f_\zeta + S\left(\bar\zeta,\bar\lambda\right) f_{\bar\lambda} \in \mathcal{D}\left(A_{\bar\gamma}\right). \tag{3.15}$$

Proof. First, let us recall that formula (2.14) provides one-to-one correspondence between $f_\lambda \in \mathfrak{N}_\lambda$ and $f_\gamma \in \mathfrak{N}_\gamma$. The same is valid for $f_\zeta \in \mathfrak{N}_\zeta$ and $f_{\bar\gamma} \in \mathfrak{N}_{\bar\gamma}$.

Let $f_\lambda \in \mathfrak{N}_\lambda$ and $f_\gamma = f_0 + f_\lambda + f_{\bar\lambda} = g_0 + g_\zeta + g_{\bar\zeta}$. If

$$g_\zeta = u_0 + u_\lambda + u_{\bar\lambda}, \qquad g_{\bar\zeta} = v_0 + v_\lambda + v_{\bar\lambda}$$

then $f_{\bar\lambda} = u_{\bar\lambda} + v_{\bar\lambda}$. Clearly $u_{\bar\lambda} = \Theta\left(\bar\lambda, \zeta\right) g_\zeta$, $v_{\bar\lambda} = \Theta\left(\bar\lambda, \bar\zeta\right) g_{\bar\zeta}$. From (2.14) it follows that $g_{\bar\zeta} = \Theta_{\bar\zeta}(\gamma) g_\zeta$, hence, taking into account (2.15), we get

$$f_{\bar\lambda} = \left[\Theta\left(\bar\lambda, \bar\zeta\right) + \Theta\left(\bar\lambda, \zeta\right)\Theta\left(\zeta, \gamma\right)\Theta^{-1}\left(\bar\zeta, \gamma\right)\right] g_\zeta = \Theta\left(\bar\lambda, \gamma\right)\Theta^{-1}\left(\bar\zeta, \gamma\right) f_{\bar\zeta}.$$

Thus we obtain

$$f_\gamma - f_0 = f_\lambda + S\left(\bar\lambda, \bar\zeta\right) g_{\bar\zeta} \in \mathcal{D}\left(A_\gamma\right).$$

Analogously, if $f_\zeta \in \mathfrak{N}_\zeta$ and $f_{\bar\gamma} = f_0 + f_\zeta + f_{\bar\zeta} = g_0 + g_{\bar\lambda} + g_\lambda$, then

$$f_{\bar\gamma} - f_0 = f_\zeta + \Theta\left(\bar\zeta, \bar\gamma\right)\Theta^{-1}\left(\bar\lambda, \bar\gamma\right) g_{\bar\lambda} \in \mathcal{D}\left(A_{\bar\gamma}\right),$$

completing the proof.

In the presence of relations (3.15), Θ-functions of A_γ and $A_{\bar\gamma}$ appeared in [7] simultaneously as a factors of the product $S\left(\bar\lambda, \bar\zeta\right)\ S\left(\bar\zeta, \bar\lambda\right)$. From (3.14) we have

$$S\left(\bar\lambda, \bar\zeta\right)\ S\left(\bar\zeta, \bar\lambda\right) = \Theta\left(\bar\lambda, \gamma\right)\Theta^{-1}\left(\bar\zeta, \gamma\right)\Theta\left(\bar\zeta, \bar\gamma\right)\Theta^{-1}\left(\bar\lambda, \bar\gamma\right) =$$

$$= \Theta\left(\bar\lambda, \bar\gamma\right)\Theta^{-1}\left(\bar\lambda, \bar\gamma\right)\Theta\left(\bar\lambda, \gamma\right)\Theta^{-1}\left(\bar\zeta, \gamma\right)\Theta\left(\bar\zeta, \bar\gamma\right)\Theta^{-1}\left(\bar\lambda, \bar\gamma\right).$$

Since (2.16) and (2.14) imply that

$$\Theta^{-1}\left(\bar\lambda, \bar\gamma\right)\Theta\left(\bar\lambda, \gamma\right) = -\Theta\left(\bar\gamma, \lambda\right)\Theta^{-1}\left(\gamma, \lambda\right) = -\Theta_\gamma(\lambda),$$

$$\Theta^{-1}\left(\bar\zeta, \gamma\right)\Theta\left(\bar\zeta, \bar\gamma\right) = -\Theta\left(\gamma, \zeta\right)\Theta^{-1}\left(\bar\gamma, \zeta\right) = -\Theta_{\bar\gamma}(\zeta),$$

hence we obtain

$$S\left(\bar\lambda, \bar\zeta\right)\ S\left(\bar\zeta, \bar\lambda\right) = \Theta\left(\bar\lambda, \bar\gamma\right)\Theta_\gamma\left(\lambda\right)\Theta_{\bar\gamma}(\zeta)\Theta^{-1}\left(\bar\lambda, \bar\gamma\right).$$

The formula obtained coincides with that in [7] up to the order of multiplication by invertible function $\Theta\left(\bar\lambda, \bar\gamma\right)$.

3.2.

Θ-function of arbitrary maximal dissipative (accumulative) extension of symmetric canonical differential operator was introduced in [8] by the Kuzhel's approach. Here we present the same formulas for arbitrary maximal dissipative extension of A_0, and for its adjoint.

Let some $\gamma \in C^+$ be fixed and $K_\gamma \in [\mathfrak{N}_{\bar\gamma}, \mathfrak{N}_\gamma]$ be a contraction. Consider extensions of A_0, defined by

$$A_{K_\gamma} = A_0^*|Ker\left(\mathcal{P}_{\bar\gamma} - K_\gamma^*\mathcal{P}_\gamma\right); \qquad A_{K_\gamma^*} = A_0^*|Ker\left(\mathcal{P}_\gamma - K_\gamma\mathcal{P}_{\bar\gamma}\right). \tag{3.16}$$

According to [5] (Sec. 3.4) formula (3.16) establishes one-to-one correspondence between the set $\{K_\gamma \in [\mathfrak{N}_{\bar\gamma}, \mathfrak{N}_\gamma]\}$ of contractions and the set $\{A_{K_\gamma}\}$ ($\{A_{K_\gamma^*} = A_{K_\gamma}^*\}$) of maximal dissipative (accumulative) extensions of A_0.

Now consider the operator functions

$$(\mathcal{P}_\gamma - K_\gamma \mathcal{P}_{\bar\gamma})\,|\mathfrak{N}_\lambda = \Theta(\gamma,\lambda) - K_\gamma \Theta(\bar\gamma,\lambda)\,, \quad \left(\mathcal{P}_{\bar\gamma} - K_\gamma^* \mathcal{P}_\gamma\right)|\mathfrak{N}_\lambda = \Theta(\bar\gamma,\lambda) - K_\gamma^*\Theta(\gamma,\lambda)\,;$$

$$\left(\mathcal{P}_{\bar\gamma} - K_\gamma^* \mathcal{P}_\gamma\right)|\mathfrak{N}_\zeta = \Theta(\bar\gamma,\zeta) - K_\gamma^*\Theta(\gamma,\zeta)\,, \quad (\mathcal{P}_\gamma - K_\gamma \mathcal{P}_{\bar\gamma})\,|\mathfrak{N}_\zeta = \Theta(\gamma,\zeta) - K_\gamma\Theta(\bar\gamma,\zeta)\,.$$

Proposition 2.8 enables to introduce analytical in C^+, C^- operator functions

$$\begin{aligned} \Theta_{K_\gamma}(\lambda) &= \left[\Theta(\bar\gamma,\lambda) - K_\gamma^*\Theta(\gamma,\lambda)\right]\left[\Theta(\gamma,\lambda) - K_\gamma\Theta(\bar\gamma,\lambda)\right]^{-1} = \\ &= \left[\Theta_\gamma(\lambda) - K_\gamma^*\right]\left[I_\gamma - K_\gamma\Theta_\gamma(\lambda)\right]^{-1}, \\ \Theta_{K_\gamma^*}(\zeta) &= \left[\Theta(\gamma,\zeta) - K_\gamma\Theta(\bar\gamma,\zeta)\right]\left[\Theta(\bar\gamma,\zeta) - K_\gamma^*\Theta(\gamma,\zeta)\right]^{-1} = \\ &= \left[\Theta_{\bar\gamma}(\zeta) - K_\gamma\right]\left[I_{\bar\gamma} - K_\gamma^*\Theta_\gamma(\zeta)\right]^{-1}, \end{aligned} \tag{3.17}$$

as Θ-functions of A_{K_γ} and $A_{K_\gamma^*}$ respectively.

Proposition 3.4 *The operator $\Theta_{K_\gamma}(\lambda_0)$ is not invertible if and only if $\lambda_0 \in \sigma_p\left(A_{K_\gamma}\right)$.*

Proof. Let $f_{\lambda_0} \in \mathfrak{N}_{\lambda_0}$ and $\left[\Theta(\bar\gamma,\lambda_0) - K_\gamma^*\Theta(\gamma,\lambda_0)\right] f_{\lambda_0} = 0$. Then $\left(\mathcal{P}_{\bar\gamma} - K_\gamma^*\mathcal{P}_\gamma\right) f_{\lambda_0} = 0$, hence $f_{\lambda_0} \in \mathcal{D}\left(A_{K_\gamma}\right)$ and $A_{K_\gamma} f_{\lambda_0} = A_0^* f_{\lambda_0} = \lambda_0 f_{\lambda_0}$.

If $f \in \mathcal{D}\left(A_{K_\gamma}\right)$ and $A_{K_\gamma} f = \lambda_0 f$, then $f \in \mathfrak{N}_{\lambda_0}$, since $A_{K_\gamma} f = A_0^* f$. Therefore $\left(\mathcal{P}_{\bar\gamma} - K_\gamma^*\mathcal{P}_\gamma\right) f = \left[\Theta(\bar\gamma,\lambda_0) - K^*\Theta(\gamma,\lambda_0)\right] f = 0$, and the proof is complete.

The analogous assertion is true for Θ-function of $A_{K_\gamma^*}$.

Clearly, if $K_\gamma = 0$ we have the case of extension A_γ. From Proposition 3.2 we know that Θ-function of A_γ satisfies the condition $\Theta_\gamma(\gamma) = 0$. Now we can state that such a condition is distinctive for maximal dissipative extension to be extension A_γ. Indeed, from (3.17) it follows that $\Theta_{K_\gamma}(\gamma) = -K_\gamma^*$, hence the condition $\Theta_{K_\gamma}(\gamma) = 0$ means that $A_{K_\gamma} = A_\gamma$. Moreover, all the extensions of type (2.2) can be described in terms of Θ-function (3.17).

Proposition 3.5 *Let $\varphi \neq \gamma$ be arbitrary in C^+ and $K_\gamma = \Theta_\gamma^*(\varphi)$ so that $\Theta_{\Theta_\gamma^*(\varphi)}(\varphi) = 0$. Then*

$$\Theta(\bar\varphi,\bar\gamma)\,\Theta_{\Theta_\gamma^*(\varphi)}(\lambda) = \Theta_\varphi(\lambda)\,\Theta(\varphi,\gamma)\,.$$

Proof. From (3.17) we have

$$\Theta_{\Theta_\gamma^*(\varphi)}(\lambda) = \left[\Theta(\bar\gamma,\lambda) - \Theta_\gamma(\varphi)\,\Theta(\gamma,\lambda)\right]\left[\Theta(\gamma,\lambda) - \Theta_\gamma^*(\varphi)\,\Theta(\bar\gamma,\lambda)\right]^{-1}. \tag{3.18}$$

We know that $\Theta_\gamma(\varphi) = \Theta_{\bar\gamma}^*(\bar\varphi) = \Theta^{-*}(\bar\gamma,\bar\varphi)\,\Theta^*(\gamma,\bar\varphi)$, hence the first factor of (3.18) is

$$\Theta(\bar\gamma,\lambda) - \Theta^{-*}(\bar\gamma,\bar\varphi)\,\Theta^*(\gamma,\bar\varphi)\,\Theta(\gamma,\lambda) = \Theta^{-*}(\bar\gamma,\bar\varphi)\left[\Theta^*(\bar\gamma,\bar\varphi)\,\Theta(\bar\gamma,\lambda) - \Theta^*(\gamma,\bar\varphi)\,\Theta(\gamma,\lambda)\right].$$

From (2.17) and (2.15) it then follows that square bracket above is

$$\frac{Im\,\bar\varphi}{Im\,\bar\gamma}\,\Theta(\bar\varphi,\bar\gamma)\,\Theta(\bar\gamma,\lambda) - \frac{Im\,\bar\varphi}{Im\,\gamma}\,\Theta(\bar\varphi,\gamma)\,\Theta(\gamma,\lambda) = \frac{Im\,\bar\varphi}{Im\,\bar\gamma}\,\Theta(\bar\varphi,\lambda)\,,$$

thus we get

$$\Theta(\bar{\gamma},\lambda)-\Theta_{\gamma}(\varphi)\,\Theta(\gamma,\lambda)=\frac{Im\,\bar{\varphi}}{Im\,\bar{\gamma}}\,\Theta^{-*}(\bar{\gamma},\bar{\varphi})\,\Theta(\bar{\varphi},\lambda)\,.$$

By similar computations we have

$$\Theta(\gamma,\lambda)-\Theta_{\gamma}^{*}(\varphi)\,\Theta(\bar{\gamma},\lambda)=\frac{Im\,\varphi}{Im\,\gamma}\,\Theta^{-*}(\gamma,\varphi)\,\Theta(\varphi,\lambda)\,,$$

hence formula (3.18) takes the form

$$\Theta_{\Theta_{\gamma}^{*}(\varphi)}(\lambda)=\Theta^{-*}(\bar{\gamma},\bar{\varphi})\,\Theta(\bar{\varphi},\lambda)\,\Theta^{-1}(\varphi,\lambda)\,\Theta^{*}(\gamma,\varphi)=\Theta^{-*}(\bar{\gamma},\bar{\varphi})\,\Theta_{\varphi}(\lambda)\,\Theta^{*}(\gamma,\varphi)\,.$$

Finally, applying equation (2.17) once more, we arrive at the result

$$\Theta_{\Theta_{\gamma}^{*}(\varphi)}(\lambda)=\Theta^{-1}(\bar{\varphi},\bar{\gamma})\,\Theta_{\varphi}(\lambda)\,\Theta(\varphi,\gamma)\,.$$

4. On Weyl function and some of its applications

4.1.

Let $\gamma_0\in C^{+}$ be fixed and $V(\gamma_0)\in[\mathfrak{N}_{\bar{\gamma}_0},\mathfrak{N}_{\gamma_0}]$ be an isometry. Consider the von Neumann's self-adjoint extensions

$$\mathcal{D}(A_{\pm V})=Ker\,[\mathcal{P}_{\gamma_0}\mp V(\gamma_0)\mathcal{P}_{\bar{\gamma}_0}]\,;\quad A_{\pm V}=A^{*}|\mathcal{D}(A_{\pm V}) \tag{4.1}$$

of A_0. Let $\gamma\neq\gamma_0$ be arbitrary. Then, by Proposition 2.8, the operators

$$V_{\pm}^{*}(\gamma)=\Phi_{W(\gamma,\gamma_0)}[\mp V^{*}(\gamma_0)]\in[\mathfrak{N}_{\gamma},\mathfrak{N}_{\bar{\gamma}}]$$

are isometries too. Making use (2.20) and (2.18) one can see that

$$V_{\pm}(\gamma)=\Phi_{W(\gamma,\gamma_0)}^{*}[\mp V^{*}(\gamma_0)]=-M_{\gamma_0\pm}^{-1}(\gamma)M_{\gamma_0\pm}(\bar{\gamma})\in[\mathfrak{N}_{\bar{\gamma}},\mathfrak{N}_{\gamma}]\,, \tag{4.2}$$

where

$$M_{\gamma_0\pm}(\gamma)=\Theta(\gamma_0,\gamma)\pm V(\gamma_0)\Theta(\bar{\gamma}_0,\gamma)\,. \tag{4.3}$$

Evidently $M_{\gamma_0\pm}(\gamma_0)=I_{\gamma_0}$, $M_{\gamma_0\pm}(\bar{\gamma}_0)=\pm V(\gamma_0)$.

The following assertion establishes the meaning of isometries $V_{\pm}(\gamma)$.

Proposition 4.1 *Let self-adjoint extensions $A_{\pm V}$ be given by* (4.1). *Then the isometries $V_{\pm}(\gamma)$ defined by* (4.2), (4.3) *are such that*

$$Ker\,[\mathcal{P}_{\gamma}-V_{\mp}(\gamma)\mathcal{P}_{\bar{\gamma}}]=Ker\,[\mathcal{P}_{\gamma_0}\mp V(\gamma_0)\mathcal{P}_{\bar{\gamma}_0}]=\mathcal{D}(A_{\pm V})$$

for arbitrary $\gamma\in C^{+}$.

Proof. Since $f = f_0 + f_\gamma + f_{\bar\gamma} \in \mathcal{D}(A_V)$ if and only if $[\mathcal{P}_{\gamma_0} - V(\gamma_0)\mathcal{P}_{\bar\gamma_0}]\, f = 0$, therefore, in view of (4.3), we have

$$[\mathcal{P}_{\gamma_0} - V(\gamma_0)\mathcal{P}_{\bar\gamma_0}]\, f_\gamma + [\mathcal{P}_{\gamma_0} - V(\gamma_0)\mathcal{P}_{\bar\gamma_0}]\, f_{\bar\gamma} = M_{\gamma_0 -}(\gamma) f_\gamma + M_{\gamma_0 -}(\bar\gamma) f_{\bar\gamma} = 0.$$

Thus we obtain $f_\gamma + M^{-1}_{\gamma_0 -}(\gamma) M_{\gamma_0 -}(\bar\gamma) f_{\bar\gamma} = 0$, or, due to (4.2), $f \in Ker\,[\mathcal{P}_\gamma - V_-(\gamma)\mathcal{P}_{\bar\gamma}]$.

The case of A_{-V} and $V_+(\gamma)$ is treated analogously, completing the proof.

We wish to emphasize that formulas (4.2) and (4.3) build the family $\{V_-(\gamma)\}_{\gamma \in C^+}$ of isometries, which is associated with A_V (or $V(\gamma_0)$) in the sense of Proposition 4.1.

Following V. A. Derkach, M. M. Malamud [1] (Sec 1), the operator-valued function

$$M_V(\varphi) = i M_{\gamma_0 +}(\varphi) M^{-1}_{\gamma_0 -}(\varphi), \quad \varphi \in C^- \cup C^+, \tag{4.4}$$

analytical in $C^- \cup C^+$, is the Weyl function of A_V. The Weyl function of A_{-V} is

$$M_{-V}(\varphi) = i M_{\gamma_0 -}(\varphi) M^{-1}_{\gamma_0 +}(\varphi) = -M_V^{-1}(\varphi).$$

Proposition 4.2 *The Weyl function satisfies the identity*

$$M_V(\varphi) - M_V^*(\psi) = 2i \frac{Im\,\psi}{Im\,\gamma_0}\, M^{-*}_{\gamma_0 -}(\psi)\Theta(\psi,\varphi) M^{-1}_{\gamma_0 -}(\varphi). \tag{4.5}$$

Proof. From definition (4.4) we have

$$M_V(\varphi) - M_V^*(\psi) = M^{-*}_{\gamma_0 -}(\psi) \left[M^*_{\gamma_0 -}(\psi) M_{\gamma_0 +}(\varphi) + M^*_{\gamma_0 +}(\psi) M_{\gamma_0 -}(\varphi) \right] M^{-1}_{\gamma_0 -}(\varphi).$$

Taking into account (4.3), (2.18) and (2.16), the square bracket above is readily transformed to

$$2\left[\Theta^*(\gamma_0,\psi)\Theta(\gamma_0,\varphi) - \Theta^*(\bar\gamma_0,\psi)\,\Theta(\bar\gamma_0,\varphi)\right] =$$
$$= 2\frac{Im\,\psi}{Im\,\gamma_0}\left[\Theta(\psi,\gamma_0)\Theta(\gamma_0,\varphi) + \Theta(\psi,\bar\gamma_0)\,\Theta(\bar\gamma_0,\varphi)\right] = 2\frac{Im\,\psi}{Im\,\gamma_0}\,\Theta(\psi,\varphi)$$

with the result.

As a corollary, setting in (4.5) first $\psi = \bar\varphi$, and then $\psi = \varphi$, we obtain the following well known properties of M-function (see [1] Sec. 1)

$$M_V(\varphi) = M_V^*(\bar\varphi); \quad \frac{M_V(\varphi) - M_V^*(\varphi)}{\varphi - \bar\varphi} = \frac{1}{Im\,\gamma_0} M^{-*}_{\gamma_0 -}(\varphi) M^{-1}_{\gamma_0 -}(\varphi). \tag{4.6}$$

From (4.4) and (4.3) it follows that in C^+ we have also the following presentation of M-function by means of Θ-function

$$\begin{aligned} M_V(\lambda) &= i\,[I_{\gamma_0} + V(\gamma_0)\Theta_{\gamma_0}(\lambda)]\,[I_{\gamma_0} - V(\gamma_0)\Theta_{\gamma_0}(\lambda)]^{-1} = \\ &= i\,[I_{\gamma_0} - V(\gamma_0)\Theta_{\gamma_0}(\lambda)]^{-1}\,[I_{\gamma_0} + V(\gamma_0)\Theta_{\gamma_0}(\lambda)] = \\ &= i\left\{I_{\gamma_0} + 2\,[I_{\gamma_0} - V(\gamma_0)\Theta_{\gamma_0}(\lambda)]^{-1}\, V(\gamma_0)\Theta_{\gamma_0}(\lambda)\right\}. \end{aligned} \tag{4.7}$$

The presentation of M-function in C^- is similar

$$M_V(\zeta) = i\left[\Theta_{\bar\gamma_0}(\zeta) + V(\gamma_0)\right]\left[\Theta_{\bar\gamma_0}(\zeta) - V(\gamma_0)\right]^{-1} =$$
$$= -i\left[I_{\gamma_0} + \Theta_{\bar\gamma_0}(\zeta)V^*(\gamma_0)\right]\left[I_{\gamma_0} - \Theta_{\bar\gamma_0}(\zeta)V^*(\gamma_0)\right]^{-1} = \tag{4.8}$$
$$= -i\left\{I_{\gamma_0} + 2\Theta_{\bar\gamma_0}(\zeta)V^*(\gamma_0)\left[I_{\gamma_0} - \Theta_{\bar\gamma_0}(\zeta)\right]^{-1}\right\}.$$

In the presence of Proposition 4.1 and formula (4.4), we have another M-function of A_V, namely, the operator function

$$M_{V_-(\gamma)}(\varphi) = i\left[\Theta(\gamma,\varphi) + V_-(\gamma)\Theta(\bar\gamma,\varphi)\right]\left[\Theta(\gamma,\varphi) - V_-(\gamma)\Theta(\bar\gamma,\varphi)\right]^{-1}. \tag{4.9}$$

Again, in C^+ it holds that

$$M_{V_-(\gamma)}(\lambda) = i\left[I_\gamma + V_-(\gamma)\Theta_\gamma(\lambda)\right]\left[I_\gamma - V_-(\gamma)\Theta_\gamma(\lambda)\right]^{-1} =$$
$$= i\left\{I_\gamma + 2\left[I_\gamma - V_-(\gamma)\Theta_\gamma(\lambda)\right]^{-1} V_-(\gamma)\Theta_\gamma(\lambda)\right].$$

Now with A_V (or $M_V(\cdot)$) associates the family $\left\{(M_V)_\gamma(\cdot) := M_{V_-(\gamma)}(\cdot)\right\}_{\gamma\in C^+}$ of M-functions of A_V. The following theorem contains its description in terms of M-function $M_V(\cdot)$.

Theorem 4.1 *For arbitrary $\gamma \in C^+$ the following formula takes place*

$$(M_V)_\gamma(\lambda) = i\left\{I - 2\Theta(\gamma,\lambda)\Theta^{-1}(\gamma_0,\lambda)\left[I_{\gamma_0} - V(\gamma_0)\Theta_{\gamma_0}(\lambda)\right]^{-1} M_{\gamma_0 -}(\bar\gamma)\,\Theta_\gamma(\lambda)\right\}. \tag{4.10}$$

Proof. From (4.2) we have

$$\left[I_\gamma - V_-(\gamma)\Theta_\gamma(\lambda)\right]^{-1} = \left[I_\gamma + M^{-1}_{\gamma_0 -}(\gamma)M_{\gamma_0 -}(\bar\gamma)\,\Theta_\gamma(\lambda)\right]^{-1} =$$
$$= \left[M_{\gamma_0 -}(\gamma) + M_{\gamma_0 -}(\bar\gamma)\,\Theta_\gamma(\lambda)\right]^{-1} M_{\gamma_0 -}(\gamma) =$$
$$= \Theta(\gamma,\lambda)\left[M_{\gamma_0 -}(\gamma)\Theta(\gamma,\lambda) + M_{\gamma_0 -}(\bar\gamma)\,\Theta(\bar\gamma,\lambda)\right]^{-1} M_{\gamma_0 -}(\gamma).$$

In view of (4.3), the square bracket above is

$$\Theta(\gamma_0,\gamma)\,\Theta(\gamma,\lambda) - V(\gamma_0)\Theta(\bar\gamma_0,\gamma)\,\Theta(\gamma,\lambda) + \Theta(\gamma_0,\bar\gamma)\,\Theta(\bar\gamma,\lambda) - V(\gamma_0)\Theta(\bar\gamma_0,\bar\gamma)\,\Theta(\bar\gamma,\lambda) =$$
$$= \Theta(\gamma_0,\lambda) - V(\gamma_0)\Theta(\bar\gamma_0,\lambda) = \left[I_{\gamma_0} - V(\gamma_0)\Theta_{\gamma_0}(\lambda)\right]\Theta(\gamma_0,\lambda).$$

Thus formula (4.9) can be rewritten as

$$M_{V_-(\gamma)}(\lambda) = i\left\{I_\gamma + 2\Theta(\gamma,\lambda)\Theta^{-1}(\gamma_0,\lambda)\left[I_{\gamma_0} - V(\gamma_0)\Theta_{\gamma_0}(\lambda)\right]^{-1} M_{\gamma_0 -}(\gamma)\,V_-(\gamma)\Theta_\gamma(\lambda)\right\},$$

which is (4.10), since $V_-(\gamma) = -M^{-1}_{\gamma_0 -}(\gamma)M_{\gamma_0 -}(\bar\gamma)$.

If $\gamma = \gamma_0$ in (4.10), apparently we have (4.8), since $M_{\gamma_0 -}(\bar\gamma_0) = -V(\gamma_0)$, and the proof is complete.

The analogous formula can be obtained for $(M_V)_\gamma(\zeta)$, $\zeta \in C^-$ by virtue of (4.9) and $M_V(\zeta) = M_V^*\left(\bar\zeta\right)$.

4.2.

Here we present modified formulas of von Neumann, stimulated by the following alteration of decomposition (2.1).

For arbitrary $\gamma \in C^+$ and $\zeta \in C^-$ it takes place the direct-sum decomposition

$$\mathcal{D}\left(A_0^*\right) = \mathcal{D}\left(A_0\right) \dot{+} \mathfrak{N}_\gamma \dot{+} \mathfrak{N}_\zeta. \tag{4.11}$$

Indeed, since $Ran(A_\gamma - \zeta I) = \mathfrak{H}$, hence for any $f \in \mathcal{D}\left(A_0^*\right)$ there exists the unique $g \in \mathcal{D}(A_\gamma)$ such that $(A_0^* - \zeta I)f = (A_\gamma - \zeta I)g$. Thus we have $(A_0^* - \zeta I)(f-g) = 0$, which is $f - g \in \mathfrak{N}_\zeta$, proving (4.11).

Let domain of self-adjoint extension A_V be given by

$$\mathcal{D}\left(A_V\right) = \left\{f \in \mathcal{D}\left(A_0^*\right);\ f = f_0 + V(\gamma)f_{\bar{\gamma}} + f_{\bar{\gamma}}\right\}.$$

In the presence of (4.11) we have

$$\mathcal{D}\left(A_V\right) \ni f = f_0 + V(\gamma)f_{\bar{\gamma}} + f_{\bar{\gamma}} = g_0 + g_\gamma + g_\zeta. \tag{4.12}$$

From (2.13) we know that

$$g_\zeta = u_0 + u_\gamma + u_{\bar{\gamma}} = u_0 + \Theta(\gamma,\zeta)g_\zeta + \Theta\left(\bar{\gamma},\zeta\right)g_\zeta,$$

therefore

$$f = f_0 + V(\gamma)f_{\bar{\gamma}} + f_{\bar{\gamma}} = (g_0 + u_0) + [g_\gamma + \Theta(\gamma,\zeta)g_\zeta] + \Theta\left(\bar{\gamma},\zeta\right)g_\zeta,$$

which means that $g_\gamma + \Theta(\gamma,\zeta)g_\zeta = V(\gamma)f_{\bar{\gamma}}, \quad f_{\bar{\gamma}} = \Theta\left(\bar{\gamma},\zeta\right)g_\zeta$.

Thus we have $g_\gamma = V(\gamma)f_{\bar{\gamma}} - \Theta(\gamma,\zeta)g_\zeta = \left[V(\gamma)\Theta\left(\bar{\gamma},\zeta\right) - \Theta\left(\gamma,\zeta\right)\right]g_\zeta$, and, in view of (4.3), formula (4.12) takes the final form

$$\mathcal{D}\left(A_V\right) \ni f = g_0 - \left[\Theta\left(\gamma,\zeta\right) - V(\gamma)\Theta\left(\bar{\gamma},\zeta\right)\right]g_\zeta + g_\zeta = g_0 - M_{\gamma-}(\zeta)g_\zeta + g_\zeta. \tag{4.13}$$

Theorem 4.2 *Let an operator $M(\gamma,\zeta) \in [\mathfrak{N}_\zeta, \mathfrak{N}_\gamma]$ be bounded invertible. Then the operator*

$$A_M = A_0^* | Ker\left[\mathcal{P}_\gamma + M(\gamma,\zeta)\mathcal{P}_\zeta\right]$$

is a self-adjoint extension of A_0 if and only if $M\left(\gamma,\zeta\right)$ possesses the property

$$\begin{gathered}\left(\gamma - \bar{\gamma}\right)M^*(\gamma,\zeta)M\left(\gamma,\zeta\right) + \left(\zeta - \bar{\zeta}\right)I_\zeta - \\ -\left(\gamma - \bar{\gamma}\right)\left[\Theta^*\left(\gamma,\zeta\right)M\left(\gamma,\zeta\right) + M^*\left(\gamma,\zeta\right)\Theta\left(\gamma,\zeta\right)\right] = 0.\end{gathered} \tag{4.14}$$

Proof. In the presence of (4.13), the necessary condition will be proved, if we will show that the function $M_{\gamma -}(\zeta)$ satisfies (4.14). By virtue of (2.17) and (2.15) we have

$$M_{\gamma -}^{*}(\zeta)M_{\gamma -}(\zeta) = \left[\Theta^{*}\left(\gamma,\zeta\right) - \Theta^{*}\left(\bar{\gamma},\zeta\right)V^{*}(\gamma)\right]\left[\Theta\left(\gamma,\zeta\right) - V(\gamma)\Theta\left(\bar{\gamma},\zeta\right)\right] =$$
$$= \left[\Theta^{*}\left(\gamma,\zeta\right)\Theta\left(\gamma,\zeta\right) + \Theta^{*}\left(\bar{\gamma},\zeta\right)\Theta\left(\bar{\gamma},\zeta\right)\right] - \left[\Theta^{*}\left(\gamma,\zeta\right)V(\gamma)\Theta\left(\bar{\gamma},\zeta\right) + \Theta^{*}\left(\bar{\gamma},\zeta\right)V^{*}(\gamma)\Theta\left(\gamma,\zeta\right)\right] =$$
$$= \left[-\Theta^{*}\left(\gamma,\zeta\right)\Theta\left(\gamma,\zeta\right) + \Theta^{*}\left(\bar{\gamma},\zeta\right)\Theta\left(\bar{\gamma},\zeta\right)\right] +$$
$$+ \left[2\Theta^{*}\left(\gamma,\zeta\right)\Theta\left(\gamma,\zeta\right) - \Theta^{*}\left(\gamma,\zeta\right)V(\gamma)\Theta\left(\bar{\gamma},\zeta\right) - \Theta^{*}\left(\bar{\gamma},\zeta\right)V^{*}(\gamma)\Theta\left(\gamma,\zeta\right)\right] =$$
$$= -\frac{Im\,\zeta}{Im\,\gamma}\,I_{\zeta} + \Theta^{*}\left(\gamma,\zeta\right)M_{\gamma -}(\zeta) + M_{\gamma -}^{*}(\zeta)\Theta\left(\gamma,\zeta\right),$$

which is (4.14).

To prove the sufficient condition we only have to verify that the operator A_M is Hermitian, since invertibility of $M\left(\gamma,\zeta\right)$ means that deficiency index of A_M is $(0,0)$. Let $f,g \in \mathcal{D}\left(A_M\right)$, so

$$A_M f = A_0 f_0 - \gamma M\left(\gamma,\zeta\right)f_{\zeta} + \zeta f_{\zeta}, \qquad A_M g = A_0 g_0 - \gamma M\left(\gamma,\zeta\right)g_{\zeta} + \zeta g_{\zeta}.$$

By straightforward computations one can obtain

$$\begin{aligned}\langle A_M f, g\rangle - \langle f, A_M g\rangle = (\gamma-\bar{\gamma})\,\langle M\left(\gamma,\zeta\right)f_{\zeta}, M\left(\gamma,\zeta\right)g_{\zeta}\rangle + \left(\zeta-\bar{\zeta}\right)\langle f_{\zeta}, g_{\zeta}\rangle - \\ -\left(\gamma-\bar{\zeta}\right)\langle M\left(\gamma,\zeta\right)f_{\zeta}, g_{\zeta}\rangle - (\zeta-\bar{\gamma})\,\langle f_{\zeta}, M\left(\gamma,\zeta\right)g_{\zeta}\rangle.\end{aligned} \tag{4.15}$$

Since $M\left(\gamma,\zeta\right) \in \left[\mathfrak{N}_{\zeta},\mathfrak{N}_{\gamma}\right]$, it is apparent that

$$\langle M\left(\gamma,\zeta\right)f_{\zeta}, M\left(\gamma,\zeta\right)g_{\zeta}\rangle = \langle M^{*}\left(\gamma,\zeta\right)M\left(\gamma,\zeta\right)f_{\zeta}, g_{\zeta}\rangle.$$

To transform the last two summands of (4.15) we apply formula (2.10) in the form $\Theta\left(\gamma,\zeta\right) = \dfrac{\zeta-\bar{\gamma}}{\gamma-\bar{\gamma}}\,P_{\gamma}|\mathfrak{N}_{\zeta}$. Then

$$\left(\gamma-\bar{\zeta}\right)\langle M\left(\gamma,\zeta\right)f_{\zeta}, g_{\zeta}\rangle = \left(\gamma-\bar{\zeta}\right)\langle M\left(\gamma,\zeta\right)f_{\zeta}, P_{\gamma}g_{\zeta}\rangle =$$
$$= \left(\gamma-\bar{\zeta}\right)\frac{\bar{\gamma}-\gamma}{\bar{\zeta}-\gamma}\langle M\left(\gamma,\zeta\right)f_{\zeta}, \Theta\left(\gamma,\zeta\right)g_{\zeta}\rangle = (\gamma-\bar{\gamma})\,\langle\Theta^{*}\left(\gamma,\zeta\right)M\left(\gamma,\zeta\right)f_{\zeta}, g_{\zeta}\rangle.$$

Analogously,

$$(\zeta-\bar{\gamma})\,\langle f_{\zeta}, M\left(\gamma,\zeta\right)g_{\zeta}\rangle = (\zeta-\bar{\gamma})\,\langle P_{\gamma}f_{\zeta}, M\left(\gamma,\zeta\right)g_{\zeta}\rangle =$$
$$= (\zeta-\bar{\gamma})\,\frac{\gamma-\bar{\gamma}}{\zeta-\bar{\gamma}}\langle\Theta\left(\bar{\gamma},\zeta\right)f_{\zeta}, M\left(\gamma,\zeta\right)g_{\zeta}\rangle = (\gamma-\bar{\gamma})\,\langle M^{*}\left(\gamma,\zeta\right)\Theta\left(\gamma,\zeta\right)f_{\zeta}, g_{\zeta}\rangle.$$

The proof is finished.

Note that if A_M is a self-adjoint extension defined by $M(\gamma,\zeta)$, then there exists the unique isometry $V_M \in \left[\mathfrak{N}_{\bar{\gamma}},\mathfrak{N}_{\gamma}\right]$ such that $A_{V_M} = A_M$. Uniqueness of presentation (4.12) and formula (4.13) imply that

$$M(\gamma,\zeta) = \Theta\left(\gamma,\zeta\right) - V_M\Theta\left(\bar{\gamma},\zeta\right).$$

The special particular case of Theorem 4.2 we get, when $\zeta = -\gamma$. Then $M_{\gamma -}(-\gamma) \in \left[\mathfrak{N}_{-\gamma},\mathfrak{N}_{\gamma}\right]$ and corresponding formulas are

$$f = f_0 - M_{\gamma -}(-\gamma)f_{-\gamma} + f_{-\gamma}; \qquad A_M f = A_0 f_0 - \gamma M_{\gamma -}(-\gamma)f_{-\gamma} - \gamma f_{-\gamma}.$$

4.3.

Let A_V be the self-adjoint extension of A_0 of preceding Section and $R(A_V, \lambda)$ be its resolvent on C^+.

Proposition 4.3 *The resolvent of* A_V *satisfies the identity*

$$R(A_V, \lambda)|\mathfrak{N}_{\bar{\lambda}} = -\frac{1}{\lambda - \bar{\lambda}}[I_{\bar{\lambda}} + V_-(\lambda)] = -\frac{1}{\lambda - \bar{\lambda}}\left[I_{\bar{\lambda}} - M_{\gamma-}^{-1}(\lambda)M_{\gamma-}\left(\bar{\lambda}\right)\right]. \tag{4.16}$$

Proof. Let $f_{\bar{\lambda}} \in \mathfrak{N}_{\bar{\lambda}}$ be arbitrary. Since

$$\left(A_V - \bar{\lambda}I\right)(A_V - \lambda I)^{-1} f_{\bar{\lambda}} = \left[I_{\bar{\lambda}} + \left(\lambda - \bar{\lambda}\right)(A_V - \lambda I)^{-1}\right] f_{\bar{\lambda}} = f_\lambda \in \mathfrak{N}_\lambda,$$

hence

$$f_\lambda - f_{\bar{\lambda}} = \left(\lambda - \bar{\lambda}\right)(A_V - \lambda I)^{-1} f_{\bar{\lambda}} = \left(\lambda - \bar{\lambda}\right) R(A_V, \lambda) f_{\bar{\lambda}} \in \mathcal{D}(A_V).$$

From Proposition 4.1 it now follows that $f_\lambda = -V_-(\lambda)f_{\bar{\lambda}}$, proving (4.16).

Relation (4.16) suggests one more proof of Kreĭn's resolvent formula. For short, here we omit some indices, so isometries $V, \tilde{V} \in [\mathfrak{N}_{\bar{\gamma}}, \mathfrak{N}_\gamma]$ define self-adjoint extensions A, $\tilde{A}$ of A_0 with M-functions

$$M(\lambda) = iM_+(\lambda) M_-^{-1}(\lambda), \qquad \tilde{M}(\lambda) = i\tilde{M}_+(\lambda)\tilde{M}_-^{-1}(\lambda).$$

Now from (4.16) we have

$$\left[R(\tilde{A}, \lambda) - R(A, \lambda)\right]|\mathfrak{N}_{\bar{\lambda}} = \frac{1}{\lambda - \bar{\lambda}}\left[\tilde{M}_-^{-1}(\lambda)\tilde{M}_-\left(\bar{\lambda}\right) - M_-^{-1}(\lambda) M_-\left(\bar{\lambda}\right)\right]. \tag{4.17}$$

To derive the desired formula from (4.17), first we assume that $1\overline{\in}\sigma_p(\tilde{V}V^*)$, which is necessary and sufficient for the pair $(A, \tilde{A})$ to be relatively prime, that is $\mathcal{D}(A)\cap\mathcal{D}(\tilde{A}) = \mathcal{D}(A_0)$. Next, from (4.3) it follows that

$$\Theta(\gamma, \lambda) = \frac{1}{2}[M_+(\lambda) + M_-(\lambda)], \qquad \Theta(\bar{\gamma}, \lambda) = \frac{1}{2}V^*[M_+(\lambda) - M_-(\lambda)],$$

therefore

$$\begin{aligned}\tilde{M}_-(\lambda) &= \frac{1}{2}\left\{[M_+(\lambda) + M_-(\lambda)] - \tilde{V}V^*[M_+(\lambda) - M_-(\lambda)]\right\} = \\ &= \frac{1}{2}\left[\left(I - \tilde{V}V^*\right)M_+(\lambda) + \left(I + \tilde{V}V^*\right)M_-(\lambda)\right] = \\ &= -\frac{i}{2}(I - U)\left[i(I - U)^{-1}(I + U) + M(\lambda)\right]M_-(\lambda),\end{aligned}$$

where $U = \tilde{V}V^* \in [\mathfrak{N}_\gamma]$ is a unitary operator.

The Cayley transform $\mathcal{T} = i(I + U)(I - U)^{-1}$ is a self-adjoint operator, possibly unbounded. For unbounded $\mathcal{T}$ it is readily verified that $i(I+U)(I-U)^{-1}f = i(I-U)^{-1}(I+U)f$ for any $f \in \mathcal{D}((I - U)^{-1})$, hence

$$\tilde{M}_-(\lambda) = -\frac{i}{2}(I - U)[M(\lambda) + \mathcal{T}]M_-(\lambda),$$

so

$$\tilde{M}_-^{-1}(\lambda)\tilde{M}_-\left(\bar{\lambda}\right) = M_-^{-1}(\lambda)\left[M(\lambda)+\mathcal{T}\right]^{-1}\left[M\left(\bar{\lambda}\right)+\mathcal{T}\right]M_-\left(\bar{\lambda}\right).$$

Now the right hand of (4.17) is

$$\frac{1}{\lambda-\bar{\lambda}}M_-^{-1}(\lambda)\left\{\left[M(\lambda)+\mathcal{T}\right]^{-1}\left[M\left(\bar{\lambda}\right)+\mathcal{T}\right]-I\right\}M_-\left(\bar{\lambda}\right)=$$

$$=\frac{1}{\lambda-\bar{\lambda}}M_-^{-1}(\lambda)\left[M(\lambda)+\mathcal{T}\right]^{-1}\left[M\left(\bar{\lambda}\right)-M\left(\lambda\right)\right]M_-\left(\bar{\lambda}\right).$$

Finally, taking into account (4.6), formula (4.17) transforms to the Kreĭn's resolvent formula

$$\left[R(\tilde{A},\lambda)-R(A,\lambda)\right]|\mathfrak{N}_{\bar{\lambda}} = -\frac{1}{Im\,\lambda}M_-^{-1}(\lambda)\left[M(\lambda)+\mathcal{T}\right]^{-1}M_-^{-*}\left(\bar{\lambda}\right).$$

4.4.

Here we turn to the resolvent of Weyl function. Let

$$M_V(\lambda) = i\left[I_\gamma + V\Theta_\gamma(\lambda)\right]\left[I_\gamma - V\Theta_\gamma(\lambda)\right]^{-1}, \qquad V\in\left[\mathfrak{N}_{\bar{\gamma}},\mathfrak{N}_\gamma\right].$$

In what follows we assume that some $\lambda \in C^+$ be fixed. Consider the linear fractional transformation $L(\sigma) = \omega = \dfrac{\sigma+i}{\sigma-i}$, which maps the complex plane Σ onto the plane Ω. Clearly, $L\left(\Sigma^+\right)$ and $L\left(\Sigma^-\right)$ are the interior and exterior of circle $|\omega| = 1$ respectively. The inverse mapping is $L^{-1}(\omega) = \sigma = -i\dfrac{1+\omega}{1-\omega}$.

Formula (4.6) implies that $Im\,M_V(\lambda) > 0$, hence the closed lower half-plane $Im\,\sigma \leqslant 0$ is a subset of resolvent set $\rho\left(M_V(\lambda)\right)$. Moreover, the following assertion is true.

Proposition 4.4 *The resolvent set of $M_V(\lambda)$ comprises the exterior of the closed disk*

$$\mathcal{D}(\sigma_\ell, r_\ell) = \left\{\sigma\in\Sigma^+;\ \left|\frac{\sigma-i}{\sigma+i}\right| \leqslant \ell,\ \ell = \left|\frac{\lambda-\gamma}{\lambda-\bar{\gamma}}\right| < 1\right\}$$

with the center $\sigma_\ell = i\dfrac{1+\ell^2}{1-\ell^2}$ and radius $r_\ell = \dfrac{2}{1-\ell^2}$.

A point $\sigma' \in \sigma_p\left(M_V\left(\lambda\right)\right)$ if and only if $\dfrac{1}{L(\sigma')} = \omega' \in \sigma_p\left(V\Theta_\gamma\left(\lambda\right)\right)$.

Proof. Let σ be arbitrary. Then

$$\begin{aligned} M_V(\lambda) - \sigma I_\gamma &= \left\{i\left[I_\gamma + V\Theta_\gamma(\lambda)\right] - \sigma\left[I_\gamma - V\Theta_\gamma(\lambda)\right]\right\}\left[I_\gamma - V\Theta_\gamma(\lambda)\right]^{-1} = \\ &= \left[-\left(\sigma-i\right)I_\gamma + \left(\sigma+i\right)V\Theta_\gamma(\lambda)\right]\left[I_\gamma - V\Theta_\gamma(\lambda)\right]^{-1} = \\ &= -\left(\sigma-i\right)\left[I_\gamma - \frac{\sigma+i}{\sigma-i}V\Theta_\gamma(\lambda)\right]\left[I_\gamma - V\Theta_\gamma(\lambda)\right]^{-1} = \\ &= \frac{2i}{1-\omega}\left[I_\gamma - \omega V\Theta_\gamma(\lambda)\right]\left[I_\gamma - V\Theta_\gamma(\lambda)\right]^{-1}. \end{aligned} \tag{4.18}$$

In the presence of estimate (3.9) we conclude that if $\dfrac{1}{|\omega|} > \ell$, then the operator $I_\gamma - \omega V \Theta_\gamma(\lambda)$ is bounded invertible, hence the same is $M_V(\lambda) - \sigma I$. Owing equation (4.18), we obtain also the second assertion, completing the proof.

Let $|\omega| = 1$, $\omega \neq 1$, so ωV is an isometry. The corresponding self-adjoint extension of A_0 and its M-function denote respectively by $A_{\omega V}$ and

$$M_{\omega V}(\lambda) = i\left[I_\gamma + \omega V \Theta(\lambda)\right]\left[I_\gamma - \omega V \Theta(\lambda)\right]^{-1}. \tag{4.19}$$

The following theorem is proved in [9] for canonical differential operator.

Theorem 4.3 *If* $\sigma \in (-\infty, \infty)$, *then*

$$\left[M_V(\lambda) - \sigma I_\gamma\right]^{-1} = -\frac{1}{\sigma^2+1}\left[M_{\omega V}(\lambda) + \sigma I_\gamma\right], \qquad \omega = L(\sigma).$$

Proof. From (4.19) by direct computations we get

$$\begin{gathered} M_{\omega V}(\lambda) + \sigma I_\gamma = M_{\omega V}(\lambda) - i\frac{1+\omega}{1-\omega} I_\gamma = \\ = i\left[\left(1 - \frac{1+\omega}{1-\omega}\right) I_\gamma + \omega\left(1 + \frac{1+\omega}{1-\omega}\right) V\Theta_\gamma(\lambda)\right]\left[I_\gamma - \omega V\Theta_\gamma(\lambda)\right]^{-1} = \\ = -2i\frac{\omega}{1-\omega}\left[I_\gamma - V\Theta_\gamma(\lambda)\right]\left[I_\gamma - \omega V\Theta_\gamma(\lambda)\right]^{-1}. \end{gathered}$$

Now, from (4.18) we obtain

$$\left[M_V(\lambda) - \sigma I_\gamma\right]^{-1} = \frac{1-\omega}{2i}\left[I_\gamma - V\Theta_\gamma(\lambda)\right]\left[I_\gamma - \omega V\Theta_\gamma(\lambda)\right]^{-1} = \frac{(1-\omega)^2}{4\omega}\left[M_{\omega V} + \sigma I_\gamma\right].$$

It is readily verified that $\dfrac{(1-\omega)^2}{4\omega} = -\dfrac{1}{\sigma^2+1}$, completing the proof.

Note that the formula proved above can be considered as a generalization of formula $M_V^{-1}(\lambda) = -M_{-V}(\lambda)$ in Section 4.1.

To compute the resolvent of $M_V(\lambda)$ outside of real axis, consider the family $\{K_\omega = \omega V,\ |\omega| < 1\}$ of contractions. According to (3.16) and (3.17), the Θ-function of dissipative extension A_{K_ω} is

$$\begin{gathered} \Theta_{\omega V}(\lambda) = \left[\Theta_\gamma(\lambda) - \bar{\omega} V^*\right]\left[I_\gamma - \omega V\Theta_\gamma(\lambda)\right]^{-1} = \\ = -V^*\left[\bar{\omega} I_\gamma - V\Theta_\gamma(\lambda)\right]\left[I_\gamma - \omega V\Theta_\gamma(\lambda)\right]^{-1} = -V^*\mathcal{X}_{\omega V}(\lambda). \end{gathered} \tag{4.20}$$

If $\ell < |\omega| < 1$, then the operator $\mathcal{X}_{\omega V}(\lambda)$ is invertible and

$$\begin{gathered} \mathcal{X}_{\omega V}^{-1}(\lambda) = \left[I_\gamma - \omega V\Theta_\gamma(\lambda)\right]\left[\bar{\omega} I_\gamma - V\Theta_\gamma(\lambda)\right]^{-1} = \\ = \omega\omega_*\left[\bar{\omega}_* I_\gamma - V\Theta_\gamma(\lambda)\right]\left[I_\gamma - \omega_* V\Theta_\gamma(\lambda)\right]^{-1} = \omega\omega_*\mathcal{X}_{\omega_* V}(\lambda), \end{gathered}$$

where $\omega_* = \dfrac{1}{\bar{\omega}}$ belongs to the ring $1 < |\omega| < \dfrac{1}{\ell}$. It is also clear, that the image of this ring under the mapping $\sigma = L^{-1}(\omega)$ is $\Sigma^+ \setminus \mathcal{D}(\sigma_\ell, r_\ell)$.

Theorem 4.4 *The following formula takes place*

$$[M_V(\lambda) - \sigma I_\gamma]^{-1} = \begin{cases} \dfrac{1}{\sigma - \bar\sigma} \left[\dfrac{\bar\sigma + i}{\sigma - i} \mathcal{X}_{\omega V}(\lambda) - I_\gamma \right], & \sigma \in \Sigma^-, \ \omega = L(\sigma) \\ \dfrac{1}{\sigma - \bar\sigma} \left[\dfrac{\bar\sigma + i}{\sigma - i} \mathcal{X}_{\omega_* V}(\lambda) - I_\gamma \right], & \sigma \in \Sigma^+ \setminus \mathcal{D}(\sigma_\ell, r_\ell), \ \omega_* = L(\sigma). \end{cases}$$

Proof. We repeat the steps of the preceding proof.

If $\sigma \in \Sigma^-$, from (4.20) we have

$$\mathcal{X}_{\omega V}(\lambda) + \frac{1-\bar\omega}{1-\omega} I_\gamma = \left[\left(\bar\omega + \frac{1-\bar\omega}{1-\omega} \right) I_\gamma - \left(1 + \omega \frac{1-\bar\omega}{1-\omega} \right) V\Theta_\gamma(\lambda) \right] [I_\gamma - \omega V \Theta_\gamma(\lambda)]^{-1} =$$

$$= \frac{1-\omega\bar\omega}{1-\omega} [I_\gamma - V\Theta_\gamma(\lambda)] \, [I_\gamma - \omega V\Theta_\gamma(\lambda)]^{-1},$$

hence from (4.18) it follows that

$$[M_V(\lambda) - \sigma I_\gamma]^{-1} = \frac{(1-\omega)^2}{2i(1-\omega\bar\omega)} \left[\mathcal{X}_{\omega V}(\lambda) + \frac{1-\bar\omega}{1-\omega} I_\gamma \right].$$

Again it is readily verified that

$$\frac{(1-\omega)^2}{2i} = \frac{2i}{(\sigma - i)^2}; \quad 1 - \omega\bar\omega = \frac{2i(\sigma - \bar\sigma)}{(\sigma - i)(\bar\sigma + i)}; \quad \frac{1-\bar\omega}{1-\omega} = -\frac{\sigma - i}{\bar\sigma + i},$$

thus

$$[M_V(\lambda) - \sigma I_\gamma]^{-1} = \frac{1}{\sigma - \bar\sigma} \frac{\bar\sigma + i}{\sigma - i} \left[\mathcal{X}_{\omega V}(\lambda) - \frac{\sigma - i}{\bar\sigma + i} I_\gamma \right],$$

proving the first equality.

The second is verified on the same way, taking into account that the point i is in $\mathcal{D}(\sigma_\ell, r_\ell)$. The proof is finished.

References

[1] V. A. Derkach, M. M. Malamud, Generalized resolvents and boundary value problems for Hermitian operators with gaps, J. Funct. Anal., 95, 1991, 1–95.

[2] N. Dunford, J. T. Schwartz, Linear operators, Part II, Interscience, New York, 1963.

[3] F. Gesztesy, K. Makarov, E. Tsekanovskii, An addendum to Kreĭn's formula, J. Math. Anal. Appl., 222, 1998, 594–606.

[4] J. Gohberg, S. Goldberg, M. A. Kaashoek, Classes of linear operators, Vol. 2, Birkhäuser Verlag, Basel, 1993.

[5] V. I. Gorbachuk, M. L. Gorbachuk, Boundary value problems for operator-differential equations, Naukova Dumka, Kiev, 1984 [Russian].

[6] M. G. Kreĭn, Ju. L. Shmulyan, On linear fractional transformations with operator coefficients, Mat. Issled., 2, No 3, 1967, 64–96 [Russian].

[7] A. Kuzhel, Characteristic functions and models of nonself-adjoint operators, Kluwer Acad. Publishers, Dordrecht, 1996.

[8] P. Ė. Melik-Adamyan, Description of characteristic functions of dissipative canonical operators, Izv NAN Armenii, Matematika [Russian]. [English translation: J. of Contemporary Mat. Anal., 39, No 5, 2004, 49–62.]

[9] P. Ė. Melik-Adamyan, Kreĭn-Saakyan and trace formulas for a pair of canonical differential operators, Comp. Anal. Oper. Theory, 1, 2007, 423–438.

[10] B. Sz.-Nagy, C. Foias, Harmonic analysis of operators in Hilbert space, North Holland Publ. Comp., Amsterdam, 1970.

[11] Sh. N. Saakyan, On the theory of resolvent of a symmetric operator with infinite defect numbers, DAN Arm. SSR, 41, No 2, 1965, 193–198 [Russian].

[12] A. V. Straus, The characteristic functions of linear operators, Izv. Akad. Nauk SSSR, Ser. Math., 24, No 1, 1960, 43–74 [Russian].

[13] A. V. Straus, On extensions and characteristic functions of symmetric operator, Izv. Akad. Nauk SSSR, Ser. Math., 32, No 1, 1968, 186–207 [Russian]. [English translation: Math USSR, Izv., 2, 1968.]