On the Canonical Calkin Reduction Operator

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Abstract

The topic of discussion here is the reduction operator of Calkin and his method of extensions of a Hermitian operator, variant from the von Neumann theory. An existence of a partial isometric reduction operator is established, which provides a link between these theories, and referred to as the canonical reduction operator. Concrete realization of discussed is applied to a canonical differential operator.

1 Introduction

The fundamental theory of extensions of a Hermitian operator in a Hilbert space is elaborated by J. von Neumann in [9], where a description of either maximal symmetric or self-adjoint extensions is presented by the well known method, based on the Cayley transformation.

In the memoir of J. Calkin [3] another approach is developed. Introducing the notion of reduction operator, such a description is given by means of special linear manifolds in the range of reduction operator, called the space of abstract boundary values.

The Calkin method, as opposed to the von Neumann theory, is directly applicable to differential operators, both ordinary and partial. A variant of such an application is presented in [4, XII. 4, XIII. 2].

The present paper concerns with the Calkin theory of extensions of a closed, densely defined Hermitian operator T in a Hilbert space $\mathfrak{H}$.

In Sec. 2 notations used are presented, and certain, necessary in the sequel, fundamentals of a reduction operator from [3] are recited in a slightly modified form. A characterization of the set of all reduction operators is due to M. Stone [3], and it is shown that the boundary

triplet (the space of boundary values), introduced in [2], [6], [7], [5; 3.4], is a subject of such a characterization.

In Sec. 3 the main result of this work is presented. Given an arbitrary reduction operator Γ, a partial isometric reduction operator $\tilde{\Gamma}$ is built, referred to as the canonical reduction operator because of its feature: projections $\mathcal{P}_\gamma$, $\mathcal{P}_{\bar{\gamma}}$ on defect subspaces $\mathfrak{N}_\gamma$, $\mathfrak{N}_{\bar{\gamma}}$ of T admit explicit presentations by means of $\tilde{\Gamma}$. On this way a connection between Calkin and von Neumann theories is established.

In Sec. 4 results of previous section are realized for symmetric canonical differential operator.

2 On the Calkin reduction operator

2.1. Let $\mathfrak{H}$ be a Hilbert space with an inner product $\langle f, g \rangle$, and T be a closed Hermitian operator with the domain $\mathcal{D}(T)$ dense in $\mathfrak{H}$, so T^* exists. Let $\gamma = \alpha + i\beta$, $\beta \neq 0$, and let $\mathfrak{N}_\gamma := Ker(T^* - \gamma I)$ be a defect subspace of T.

Following [10, p. 348], set $T_\gamma = \frac{1}{\beta}(T - \alpha I)$, so $T_i = T$, $\mathcal{D}(T_\gamma) = \mathcal{D}(T)$, $\mathcal{D}(T_\gamma^*) = \mathcal{D}(T^*)$. The graph of T_γ^*
$$Gr(T_\gamma^*) = \left\{ (f, T_\gamma^* f) \in \mathfrak{H} \oplus \mathfrak{H};\ f \in \mathcal{D}(T_\gamma^*) \right\},$$

as a closed linear manifold in the Hilbert space $\mathfrak{H} \oplus \mathfrak{H}$, is a Hilbert space itself, denoted by $\mathcal{G}_\gamma$ ($\mathcal{G} := \mathcal{G}_i$).

On the analogy with [4; XII. 4.2], endow the linear manifold $\mathcal{D}(T_\gamma^*)$ with the inner product

$$\langle f, g \rangle_\gamma := \langle f, g \rangle + \langle T_\gamma^* f, T_\gamma^* g \rangle. \tag{1}$$

Then the mapping $\mathcal{D}(T_\gamma^*) \ni f \to (f, T_\gamma^* f) \in \mathcal{G}_\gamma$ is an isometrical isomorphism, therefore $\mathcal{D}(T_\gamma^*)$ with the inner product (1) is a Hilbert space, denoted $\mathfrak{D}_\gamma$ ($\mathfrak{D} := \mathfrak{D}_i$).

It is clear from definition of T_γ that $\mathfrak{N}_\gamma, \mathfrak{N}_{\bar{\gamma}}$ are defect subspaces of T_γ at points $i, -i$ respectively, hence (see [4; XII. 4.10]) $\mathcal{D}(T)$, $\mathfrak{N}_\gamma$, $\mathfrak{N}_{\bar{\gamma}}$ are mutually orthogonal subspaces of $\mathfrak{D}_\gamma$, and

$$\mathfrak{D}_\gamma = \mathcal{D}(T) \oplus \mathfrak{N}_\gamma \oplus \mathfrak{N}_{\bar{\gamma}}. \tag{2}$$

This implies that the subspace $\mathfrak{N}_{\gamma\bar{\gamma}} = \mathfrak{N}_\gamma \oplus \mathfrak{N}_{\bar{\gamma}} \subset \mathfrak{D}_\gamma$ is a Hilbert space itself, and for any $f_{\gamma\bar{\gamma}} = f_\gamma + f_{\bar{\gamma}}$, $g_{\gamma\bar{\gamma}} = g_\gamma + g_{\bar{\gamma}} \in \mathfrak{N}_{\gamma\bar{\gamma}}$, where $f_\gamma, g_\gamma \in \mathfrak{N}_\gamma$, $f_{\bar{\gamma}}, g_{\bar{\gamma}} \in \mathfrak{N}_{\bar{\gamma}}$ it holds the equality

$$\langle f_{\gamma\bar{\gamma}}, g_{\gamma\bar{\gamma}} \rangle_\gamma = \langle f_\gamma, g_\gamma \rangle_\gamma + \langle f_{\bar{\gamma}}, g_{\bar{\gamma}} \rangle_\gamma = 2 \left[\langle f_\gamma, g_\gamma \rangle + \langle f_{\bar{\gamma}}, g_{\bar{\gamma}} \rangle \right]. \tag{3}$$

Let us specify also the following relation

$$\langle T^* f, g \rangle - \langle f, T^* g \rangle = \beta \left[\langle T_\gamma^* f, g \rangle - \langle f, T_\gamma^* g \rangle \right]. \tag{4}$$

Throughout this paper we will assume that $\dim \mathfrak{N}_\gamma = \dim \mathfrak{N}_{\bar{\gamma}} = n \leqslant \infty$.

2.2. Here we present some notations and facts from the treatise of Calkin [3] with the only distinction, namely, the Hilbert space $\mathcal{G}$ is replaced with that $\mathfrak{D}$, and a Hermitian operator T with equal defect numbers is reviewed. The terminology of [3] here is also preserved.

Decomposition (2) we rewrite in the form

$$\mathfrak{D} = \mathcal{D}(T) \oplus \mathfrak{N}_+ \oplus \mathfrak{N}_-, \quad \mathfrak{N}_\pm := \mathfrak{N}_{\pm i}, \quad \mathfrak{N} := \mathfrak{N}_+ \oplus \mathfrak{N}_-. \tag{5}$$

The basic concept of [3] is introduced in the following definition.

Definition [3; Def. 1.1, (1.2)]. *Let H be a Hilbert space with an inner product $\langle f, g \rangle_H$ and $[H]$ be the set of all linear bounded operators in H. Let Γ be a closed linear operator with the domain $\mathcal{D}(\Gamma)$ dense in $\mathfrak{D}$ and the range $\mathrm{Ran}\Gamma$ in H. The operator Γ is said to be a reduction operator for T^*, if there exists a unitary operator $W \in [H]$ such that for all $f, g \in \mathcal{D}(\Gamma)$ it holds the identity*

$$\langle T^* f, g \rangle - \langle f, T^* g \rangle = -\langle \Gamma f, W \Gamma g \rangle_H. \tag{6}$$

Corollaries of this definition are the following statements [3; Th. 1.1, Th. 1.2].

S.1. $\mathcal{D}(T) \subset \mathcal{D}(\Gamma)$ and $\mathrm{Ker}\Gamma = \mathcal{D}(T)$.

S.2. $\mathrm{Ran}\Gamma$ is dense in H.

S.3. The unitary operator W is such that $W^2 = -I$.

It is essential that the following statement [3; Th.3.7] also holds.

S.4. If $H_\pm$ are eigensubspaces of W, corresponding to its eigenvalues $\pm i$, then $\dim H_\pm = \dim \mathfrak{N}_\pm$.

A reduction operator may be either bounded or unbounded, and in what follows we will deal only with a bounded reduction operator, that is with the case

$$\mathcal{D}(\Gamma) = \mathfrak{D}, \quad \mathrm{Ran}\Gamma = H. \tag{7}$$

The following example of bounded reduction operator uses formula (5) (see [3; Th. 2.8, Th. 2.9], [5; 4. Th. 1.5], [6]).

Let $\mathcal{P}$ be the orthogonal projection in $\mathfrak{D}$ onto $\mathfrak{N}$. For any $f, g \in \mathfrak{D}$ it readily follows that

$$\langle T^* f, g \rangle - \langle f, T^* g \rangle = 2i \left[\langle f_+, g_+ \rangle - \langle f_-, g_- \rangle \right] = i \left[\langle f_+, g_+ \rangle_i - \langle f_-, g_- \rangle_i \right],$$

owing to (3). If the operator $W \in [\mathfrak{N}]$ assignees the vector $if_+ - if_-$ to arbitrary $f_+ + f_- \in \mathfrak{N}$, so $W^2 = -I_\mathfrak{N}$, $W^* = -W$, then $-iW(f_+ + f_-) = f_+ - f_-$, hence the above relation can be written as

$$\langle T^* f, g \rangle - \langle f, T^* g \rangle = \langle W \mathcal{P} f, \mathcal{P} g \rangle. \tag{8}$$

Clearly, W and $\mathcal{P}$ operators can be presented by the formula

$$W = i(\mathcal{P}_+ - \mathcal{P}_-)|\mathfrak{N}, \quad \mathcal{P} = \mathcal{P}_+ + \mathcal{P}_-,$$

where $\mathcal{P}_\pm$ are orthogonal projections in $\mathfrak{D}$ onto $\mathfrak{N}_\pm$.

Linear extensions of a Hermitian operator T are defined by means of reduction operator as follows [3; Def. 1.2, Th. 1.4].

Let L be a linear manifold in H. Then the operator

$$T_L = T^*|\mathcal{D}_L, \quad \mathcal{D}_L = \{f \in \mathcal{D}(\Gamma); \Gamma f \in L\}$$

is a linear extension of T, since, clearly, $\mathcal{D}_L$ is a linear manifold in $\mathfrak{D}$, and $\mathcal{D}(T) \subset \mathcal{D}_L$.

Conversely, if T_Υ is a linear extension of T, then $T_\Upsilon = T_{L_\Upsilon}$, where

$$L_\Upsilon = \{h \in H; h = \Gamma f, f \in \mathcal{D}(T_\Upsilon)\}.$$

The Hilbert space H is called a space of abstract boundary values, and the condition $\Gamma f \in L$ is called an abstract boundary condition, defining extension T_L.

Thus, the problem of a description of self-adjoint extensions of T is reduced to a description of appropriate subspaces in H, and it is done in terms of hypermaximal W-symmetric subspaces, introduced in [3; Def. 2.2] as subspaces L such that

$$WL = H \ominus L. \tag{9}$$

Statement S. 4 allowed to call into play an isometry $V \in [H_+, H_-]$, $V^*V = I_+$, $VV^* = I_-$ ($I_\pm$ – identity operators in $H_\pm$), and prove [3; Th. 2.2], that the formula

$$L_W = Ran(I - V) = \{h \in H; h = h_+ - Vh_+, h_+ \in H_+\} \tag{10}$$

establishes an one-to-one correspondence between the set S_W of all hypermaximal W-symmetric subspaces of H and the set $\mathcal{V}_W$ of all isometries V.

Thus, the method developed by Calkin provides a clear and accomplished way from symmetric operator T to its self-adjoint extension.

Let us complete this paragraph with the following remark.

Statement S. 3 infers that the operator $\mathcal{J} = -iW$, $\mathcal{J}^* = \mathcal{J}$, $\mathcal{J}^2 = I$, is a signature operator of a Kreĭn space H, endowed also with an indefinite inner product $[h_1, h_2] = \langle \mathcal{J}h_1, h_2\rangle_H$.

Denote

$$P_\pm = \frac{1}{2}(I \pm \mathcal{J}), \quad H_\pm = P_\pm H, \quad H = H_+ \oplus H_-, \tag{11}$$

so $P_+ + P_- = I_H$, $P_+ - P_- = \mathcal{J}$.

In the theory of Kreĭn spaces a subspace $L \subset H$ is called hypermaximal neutral if $(\mathcal{J}L)^\perp = L$ (see [1; 4.15, 4.19]). Clearly, this condition is identical to (9).

The set of all hypermaximal neutral subspaces in H is parameterized by the set of all isometries $V \in [H_+, H_-]$ (angular operators) by the formula [1; Th. 8.10]

$$L_V = \{h \in H; \; h = h_+ + V h_+, \; h_+ \in H_+\}, \tag{12}$$

which is equivalent to (10).

For the sequel the above notations are adopted.

2.3. In what follows it is convenient to use matrix representations of vectors and operators relative to various decompositions of spaces.

For instance, in the case (11) we set

$$h = \begin{bmatrix} h_+ \\ h_- \end{bmatrix}, \quad h_\pm \in H_\pm, \quad \mathcal{J} = \begin{bmatrix} I_+ & 0 \\ 0 & -I_- \end{bmatrix}.$$

With the help of an isometry $V \in [H_+, H_-]$ let us form the operators

$$P_{+V} = \frac{1}{2} \begin{bmatrix} I_+ & V^* \\ V & I_- \end{bmatrix}, \quad P_{-V} = \frac{1}{2} \begin{bmatrix} I_+ & -V^* \\ -V & I_- \end{bmatrix}, \quad P_{\pm V}^* = P_{\pm V}, \tag{13}$$

which are reciprocally orthogonal projections in H with the properties

$$P_{\pm V} = \mathcal{J} P_{\mp V} \mathcal{J}. \tag{14}$$

Thus one has another decomposition of H

$$H = H_{+V} \oplus H_{-V}, \quad H_{\pm V} = P_{\pm V} H. \tag{15}$$

Note that formula (14) means that $H_{\pm V}$ are hypermaximal neutral subspaces of H, since $\mathcal{J} P_{\pm V} H = (I - P_{\pm V})H$.

It is readily verified that the unitary operator

$$U_V = \frac{1}{\sqrt{2}} \begin{bmatrix} I_+ & -V^* \\ V & I_- \end{bmatrix}, \quad U_V^* = U_{-V} = U_V^{-1} \tag{16}$$

is the transition operator from the orthogonal decomposition (11) to that (15), and in the new representation of H the part of a signature operator, turning H into a Kreĭn space, is played by the operator

$$\mathcal{J}_V = U_V \mathcal{J} U_V^* = \begin{bmatrix} 0 & V^* \\ V & 0 \end{bmatrix}. \tag{17}$$

The following statement [3; Th. 3.3] is included in Stone's characterization of all reduction operators for T^*.

Let Γ be a reduction operator for T^* with $\mathrm{Ran}\,\Gamma = H$, and $W \in [H]$ be the associated unitary operator. Let $\tilde{H}$ be an arbitrary Hilbert space, and $\tilde{U} \in [H, \tilde{H}]$ be any isometry.

Then the operator $\tilde{\Gamma} = \tilde{U}\Gamma$ is a reduction operator for T^* with the associated unitary operator $\tilde{W} = \tilde{U}W\tilde{U}^* \in [\tilde{H}]$.

Now let us denote $\Gamma_\pm = P_\pm\Gamma \in [\mathfrak{D}, H_\pm]$, so $Ran\Gamma_\pm = H_\pm$, and, on account of $W^* = -W$, $W = i\mathcal{J} = iP_+ - iP_-$, rewrite the identity (6) in the form

$$\langle T^*f, g\rangle - \langle f, T^*g\rangle = i\left[\langle \Gamma_+f, \Gamma_+f\rangle_H - \langle \Gamma_-f, \Gamma_-g\rangle_H\right]. \tag{18}$$

From (16) one has

$$\tilde{\Gamma}f = U_V\Gamma f = U_V\begin{bmatrix}\Gamma_+f\\\Gamma_-f\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}(\Gamma_+ - V^*\Gamma_-)f\\V(\Gamma_+ + V^*\Gamma_-)f\end{bmatrix} =: \begin{bmatrix}\Gamma_1f\\V\Gamma_2f\end{bmatrix},$$

and (17) infers that $\tilde{W} = i\mathcal{J}_V$, hence

$$-\langle \tilde{\Gamma}f, \tilde{W}\tilde{\Gamma}g\rangle_H = -\langle\begin{bmatrix}\Gamma_1f\\V\Gamma_2f\end{bmatrix}, i\begin{bmatrix}\Gamma_2\,g\\V\Gamma_1\,g\end{bmatrix}\rangle_H = i\left[\langle \Gamma_1f, \Gamma_2\,g\rangle_H + \langle \Gamma_2f, \Gamma_1\,g\rangle_H\right] =$$

$$\langle i\Gamma_1f, \Gamma_2\,g\rangle_H - \langle \Gamma_2f, i\Gamma_1\,g\rangle_H = \langle \tilde{\Gamma}_1f, \tilde{\Gamma}_2\,g\rangle_H - \langle \tilde{\Gamma}_2f, \tilde{\Gamma}_1\,g\rangle_H,$$

where

$$\tilde{\Gamma}_1 = \frac{i}{\sqrt{2}}\left(\Gamma_+ - V^*\Gamma_-\right), \quad \tilde{\Gamma}_2 = \frac{1}{\sqrt{2}}\left(\Gamma_+ + V^*\Gamma_-\right).$$

Since $Ran\Gamma_\pm = H_\pm$, with the help of formulas above it is not difficult to be convinced that operators $\tilde{\Gamma}_{1,2} \in [\mathfrak{D}, H_+]$ satisfy conditions $Ran\tilde{\Gamma}_{1,2} = H_+$, and, that for any pair $h_{+1}, h_{+2} \in H_+$ there exists $f \in \mathfrak{D}$ such that $\tilde{\Gamma}_1f = h_{+1}, \tilde{\Gamma}_2f = h_{+2}$.

Thus formula (6) admits the presentation

$$\langle T^*f, g\rangle - \langle f, T^*g\rangle = \langle \tilde{\Gamma}_1f, \tilde{\Gamma}_2\,g\rangle_{H_+} - \langle \tilde{\Gamma}_2f, \tilde{\Gamma}_1\,g\rangle_{H_+},$$

which, along with the properties of $\tilde{\Gamma}_{1,2}$ indicated above, is introduced in [2], [6], [7], [5; 3.4] as a definition of a boundary triplet $\{\tilde{\Gamma}_1, \tilde{\Gamma}_2, H_+\}$.

In conclusion of this review we refer back to formula (12). By means of projections $P_{\pm V}$ of (13) it can be presented as

$$L_V = \{h \in H;\ h = P_{+V}h\} = \{h \in H;\ P_{-V}h = 0\},$$

and the boundary condition $\Gamma f \in L_V$, defining a self-adjoint extension of T will be used in the form

$$P_{-V}\Gamma f = 0. \tag{19}$$

3 Canonical reduction operator

3.1. From now on we assume that $\operatorname{Im}\gamma = \beta > 0$, and T_γ be as in p. 2.1.

Let $\mathcal{D}'$ be any subspace of a linear space $\mathcal{D}(T^*) = \mathcal{D}(T_\gamma^*)$, and $\mathcal{D}' \supset \mathcal{D}(T) = \mathcal{D}(T_\gamma)$. Obviously, if $T' = T^*|\mathcal{D}'$, then $T_\gamma' = T_\gamma^*|\mathcal{D}' = \frac{1}{\beta}(T' - \alpha I)$, and conversely, if $T_\gamma' = T_\gamma^*|\mathcal{D}'$, then $T' = T^*|\mathcal{D}' = \beta T_\gamma' + \alpha I$, hence descriptions of extensions of T and T_γ are identical. Thus reduction operator for T_γ^* serves as that for T^* as well. Definition of reduction operator $\Gamma \in [\mathfrak{D}_\gamma, H]$ for T_γ^* by formula (6) can be written as

$$\langle T_\gamma^* f, g\rangle - \langle f, T_\gamma^* g\rangle = i\langle \mathcal{J}\Gamma f, \Gamma g\rangle_H = \frac{1}{\beta}\left[\langle T^* f, g\rangle - \langle f, T^* g\rangle\right],$$

taking into account (4) and $W = i\mathcal{J}$. Thus, in view of (18) we have

$$\langle T^* f, g\rangle - \langle f, T^* g\rangle = i\beta\left[\langle \Gamma_+ f, \Gamma_+ g\rangle_H - \langle \Gamma_- f, \Gamma_- g\rangle_H\right]. \tag{1}$$

In what follows a reduction operator for T^* shall be used in the form (1) and will be denoted $\{\Gamma, H\} = \{\Gamma_\pm, H_\pm\}$.

3.2. Let $\{\Gamma, H\} = \{\Gamma_\pm, H_\pm\}$ be a reduction operator for T^*, and let $\mathfrak{N}_{\gamma\bar\gamma}$ be the Hilbert space of p. 2.1 with the inner product inherited from $\mathfrak{D}_\gamma$.

Consider the operator $\Gamma_{\gamma\bar\gamma} = \Gamma|\mathfrak{N}_{\gamma\bar\gamma}$. From S. 1 and (7) we have $Ker\Gamma = \mathcal{D}(T)$, $Ran\Gamma = H$, hence in view of decomposition (2) it follows that $\Gamma_{\gamma\bar\gamma} \in [\mathfrak{N}_{\gamma\bar\gamma}; H]$ is bounded invertible. The operator $\Gamma_{\gamma\bar\gamma}^* \in [H; \mathfrak{N}_{\gamma\bar\gamma}]$, adjoint to $\Gamma_{\gamma\bar\gamma}$, is defined as

$$\langle \Gamma_{\gamma\bar\gamma} f, h\rangle_H = \langle f, \Gamma_{\gamma\bar\gamma}^* h\rangle_\gamma. \tag{2}$$

Consider also the operator

$$\mathcal{J}_\gamma = \begin{bmatrix} I_\gamma & 0 \\ 0 & -I_{\bar\gamma} \end{bmatrix} \in [\mathfrak{N}_{\gamma\bar\gamma}], \quad I_\gamma, I_{\bar\gamma} - \text{identity operators in } \mathfrak{N}_\gamma, \mathfrak{N}_{\bar\gamma},$$

so

$$\mathcal{J}_\gamma = P_{\gamma+} - P_{\gamma-}, \quad P_{\gamma\pm} = \frac{1}{2}(I_{\gamma\bar\gamma} \pm \mathcal{J}_\gamma), \quad I_{\gamma\bar\gamma} - \text{identity in } \mathfrak{N}_{\gamma\bar\gamma}. \tag{3}$$

Proposition 3.1 *Let* $\{\Gamma, H\} = \{\Gamma_\pm; H_\pm\}$ *be a reduction operator for* T^*, *and let* $\Gamma_{\gamma\bar\gamma} = \Gamma|\mathfrak{N}_{\gamma\bar\gamma}$. *Then*

$$\Gamma_{\gamma\bar\gamma}^* \mathcal{J}\Gamma_{\gamma\bar\gamma} = \mathcal{J}_\gamma. \tag{4}$$

Proof. Introduce into considerations the following operators

$$\theta_+(\gamma) = \Gamma_+|\mathfrak{N}_\gamma \in [\mathfrak{N}_\gamma, H_+], \quad \theta_+(\bar\gamma) = \Gamma_-|\mathfrak{N}_{\bar\gamma} \in [\mathfrak{N}_{\bar\gamma}, H_-],$$

$$\theta_-(\gamma) = \Gamma_-|\mathfrak{N}_\gamma \in [\mathfrak{N}_\gamma, H_-], \quad \theta_-(\bar\gamma) = \Gamma_+|\mathfrak{N}_{\bar\gamma} \in [\mathfrak{N}_{\bar\gamma}, H_+], \tag{5}$$

by means of which the operators $\Gamma_{\gamma\bar\gamma} \in [\mathfrak{N}_\gamma \oplus \mathfrak{N}_{\bar\gamma}, H_+ \oplus H_-]$, $\Gamma_{\gamma\bar\gamma}^* \in [H_+ \oplus H_-, \mathfrak{N}_\gamma \oplus \mathfrak{N}_{\bar\gamma}]$ admit matrix representations

$$\Gamma_{\gamma\bar\gamma} = \begin{bmatrix} \theta_+(\gamma) & \theta_-(\bar\gamma) \\ \theta_-(\gamma) & \theta_+(\bar\gamma) \end{bmatrix}, \quad \Gamma_{\gamma\bar\gamma}^* = \begin{bmatrix} \theta_+^*(\gamma) & \theta_-^*(\gamma) \\ \theta_-^*(\bar\gamma) & \theta_+^*(\bar\gamma) \end{bmatrix}. \tag{6}$$

The second formula above is owing to (2).

Identity (1) applied sequentially to pairs (f_γ, g_γ), $(f_\gamma, g_{\bar\gamma})$, $(f_{\bar\gamma}, g_\gamma)$, $(f_{\bar\gamma}, g_{\bar\gamma})$, and adjusted with (3) yields

$$(\gamma - \bar\gamma)\langle f_\gamma, g_\gamma\rangle = i\beta\langle f_\gamma, g_\gamma\rangle_\gamma = i\beta\left[\langle \theta_+(\gamma)f_\gamma, \theta_+(\gamma)g_\gamma\rangle_H - \langle \theta_-(\gamma)f_\gamma, \theta_-(\gamma)g_\gamma\rangle_H\right],$$
$$0 = \langle \theta_+(\gamma)f_\gamma, \theta_-(\bar\gamma)g_{\bar\gamma}\rangle_H - \langle \theta_-(\gamma)f_\gamma, \theta_+(\bar\gamma)g_{\bar\gamma}\rangle_H$$
$$0 = \langle \theta_-(\bar\gamma)f_{\bar\gamma}, \theta_+(\gamma)g_\gamma\rangle_H - \langle \theta_+(\bar\gamma)f_{\bar\gamma}, \theta_-(\gamma)g_\gamma\rangle_H$$
$$(\bar\gamma - \gamma)\langle f_{\bar\gamma}, g_{\bar\gamma}\rangle = -i\beta\langle f_{\bar\gamma}, g_{\bar\gamma}\rangle_\gamma = i\beta\left[\langle \theta_-(\bar\gamma)f_{\bar\gamma}, \theta_-(\bar\gamma)g_{\bar\gamma}\rangle_H - \langle \theta_+(\bar\gamma)f_{\bar\gamma}, \theta_+(\bar\gamma)g_{\bar\gamma}\rangle_H\right].$$

Because of vectors f_γ, $f_{\bar\gamma}$, g_γ, $g_{\bar\gamma}$ are arbitrary, these relations can be rewritten as

$$\begin{aligned} \theta_+^*(\gamma)\theta_+(\gamma) - \theta_-^*(\gamma)\theta_-(\gamma) &= I_\gamma \\ \theta_-^*(\bar\gamma)\theta_+(\gamma) - \theta_+^*(\bar\gamma)\theta_-(\gamma) &= 0 \\ \theta_+^*(\gamma)\theta_-(\bar\gamma) - \theta_-^*(\gamma)\theta_+(\bar\gamma) &= 0 \\ \theta_-^*(\bar\gamma)\theta_-(\bar\gamma) - \theta_+^*(\bar\gamma)\theta_+(\bar\gamma) &= -I_{\bar\gamma}. \end{aligned} \tag{7}$$

Clearly, formula (7) is an expanding transcription of (4), written in a matrix form. The proof is complete.

From (7) it also follows that the operators $\theta_+(\gamma)$, $\theta_+(\bar\gamma)$ have bounded inverses.

Formula (4) implies that $\Gamma_{\gamma\bar\gamma}^{-1} = \mathcal{J}_\gamma\Gamma_{\gamma\bar\gamma}^*\mathcal{J}$, hence

$$\Gamma_{\gamma\bar\gamma}\mathcal{J}_\gamma\Gamma_{\gamma\bar\gamma}^* = \mathcal{J}. \tag{8}$$

Proposition 3.2 *Let*

$$H_\gamma := \Gamma_{\gamma\bar\gamma}\mathfrak{N}_\gamma, \quad H_{\bar\gamma} := \Gamma_{\gamma\bar\gamma}\mathfrak{N}_{\bar\gamma}, \quad H = H_\gamma \dotplus H_{\bar\gamma}, \tag{9}$$

and let projections $P_{\gamma\pm}$ be given by formula (3).

Then the operators

$$P_\gamma = \Gamma_{\gamma\bar\gamma}P_{\gamma+}\Gamma_{\gamma\bar\gamma}^*\mathcal{J}, \quad P_{\bar\gamma} = -\Gamma_{\gamma\bar\gamma}P_{\gamma-}\Gamma_{\gamma\bar\gamma}^*\mathcal{J} \tag{10}$$

are disjoint skew projections onto subspaces H_γ, $H_{\bar\gamma}$ respectively.

Proof. First let us note that the direct sum decomposition (9) is owing to invertibility of $\Gamma_{\gamma\bar\gamma}$.

Obviously $P_\gamma + P_{\bar\gamma} = \Gamma_{\gamma\bar\gamma}\mathcal{J}_\gamma\Gamma_{\gamma\bar\gamma}^*\mathcal{J} = I$, since (8) holds.

Further, $P_\gamma H_\gamma = \Gamma_{\gamma\bar\gamma}P_{\gamma+}\Gamma_{\gamma\bar\gamma}^*\mathcal{J}\Gamma_{\gamma,\gamma}\mathfrak{N}_\gamma = \Gamma_{\gamma\bar\gamma}P_{\gamma+}\mathcal{J}_\gamma\mathfrak{N}_\gamma = \Gamma_{\gamma\bar\gamma}P_{\gamma+}\mathfrak{N}_\gamma = \Gamma_{\gamma\bar\gamma}\mathfrak{N}_\gamma = H_\gamma$, since (4) holds, $P_{\gamma+}\mathcal{J}_\gamma = P_{\gamma+}$, and $P_{\gamma+}\mathfrak{N}_\gamma = \mathfrak{N}_\gamma$.

Likewise, $P_{\bar\gamma}H_{\bar\gamma} = H_{\bar\gamma}$.

Owing to (4), (8) we have $P_\gamma^2 = \Gamma_{\gamma\bar\gamma}P_{\gamma+}\Gamma_{\gamma\bar\gamma}^*\mathcal{J}\Gamma_{\gamma\bar\gamma}P_{\gamma+}\Gamma_{\gamma\bar\gamma}^*\mathcal{J} = \Gamma_{\gamma\bar\gamma}P_{\gamma+}\mathcal{J}_\gamma P_{\gamma+}\Gamma_{\gamma\bar\gamma}^*\mathcal{J} = \Gamma_{\gamma\bar\gamma}P_{\gamma+}\Gamma_{\gamma\bar\gamma}^*\mathcal{J} = P_\gamma$, and similarly, $P_{\bar\gamma}^2 = P_{\bar\gamma}$.

Relations $P_\gamma P_{\bar\gamma} = P_{\bar\gamma} P_\gamma = 0$ are verified in the same way, completing the proof.

3.3. The following operator will play an important part for the rest of this section. Put

$$\Delta := \mathcal{J}\Gamma_{\gamma\bar\gamma}\Gamma_{\gamma\bar\gamma}^*\mathcal{J}, \quad \Delta \in [H], \quad \Delta > 0, \tag{11}$$

so the operators Δ^{-1}, $\Delta^{\frac{1}{2}}$, $\Delta^{-\frac{1}{2}}$ exist. From (4), (8) it is readily seen that $\Delta\mathcal{J}\Delta = \mathcal{J}$, or $\mathcal{J}\Delta = \Delta^{-1}\mathcal{J}$. Thus we have $\mathcal{J}\Delta^n = \Delta^{-n}\mathcal{J}$ for any natural number n, hence (see [12; 1.31])

$$\mathcal{J}\Delta^{\frac{1}{2}} = \Delta^{-\frac{1}{2}}\mathcal{J}. \tag{12}$$

Let $\tilde{H}$ be the Hilbert space of vectors in a linear space H with the inner product

$$\langle h_1, h_2\rangle_{\tilde{H}} := \langle \Delta h_1, h_2\rangle_H.$$

Introduce the inclusion operator $\mathcal{I} : H \to \tilde{H}$, which to any vector of the Hilbert space H assignees itself as a vector of $\tilde{H}$. Clearly, $\mathcal{I}^{-1} : \tilde{H} \to H$ is the inclusion operator of $\tilde{H}$ onto H, so $\mathcal{I}^{-1}\mathcal{I} = I_H$, $\mathcal{I}\mathcal{I}^{-1} = I_{\tilde{H}}$. We wish to note that the use of inclusion operators $\mathcal{I}$, $\mathcal{I}^{-1}$ also eliminates ambiguities, when a certain operator or vector in a linear space H is considered in different Hilbert spaces H and $\tilde{H}$. If $h \in H$, $B \in [H]$ notations $\mathcal{I}h =: \tilde{h}$, $\mathcal{I}B\mathcal{I}^{-1} =: \tilde{B} \in [\tilde{H}]$ will be used, so $\mathcal{I}B = \tilde{B}\mathcal{I}$, $B\mathcal{I}^{-1} = \mathcal{I}^{-1}\tilde{B}$.

With these notations the inner product defined above can be written as

$$\langle \tilde{h}_1, \tilde{h}_2\rangle_{\tilde{H}} = \langle \mathcal{I}h_1, \mathcal{I}h_2\rangle_{\tilde{H}} = \langle \Delta h_1, h_2\rangle_H. \tag{13}$$

Proposition 3.3 *The Hilbert space $\tilde{H}$ admits orthogonal decomposition*

$$\tilde{H} = \tilde{H}_+ \oplus \tilde{H}_-; \quad \tilde{H}_\pm = \tilde{P}_\pm\tilde{H}, \quad \tilde{P}_+ = \mathcal{I}P_\gamma\mathcal{I}^{-1}, \quad \tilde{P}_- = \mathcal{I}P_{\bar\gamma}\mathcal{I}^{-1}, \tag{14}$$

where P_γ, $P_{\bar\gamma}$ are defined by formula (10).

Proof. Having Proposition 3.2 proven, it is suffice to show that projections $\tilde{P}_\pm$ are self-adjoint.

To this end consider the relation

$$\langle \tilde{B}\tilde{h}_1, \tilde{h}_2\rangle_{\tilde{H}} = \langle \tilde{h}_1, \tilde{B}^*\tilde{h}_2\rangle_{\tilde{H}}, \tag{15}$$

defining adjoint operator of some $\tilde{B} \in [\tilde{H}]$. From (13) it is readily seen that the left hand of (15) is $\langle \Delta Bh_1, h_2\rangle_H$, $B = \mathcal{I}^{-1}\tilde{B}\mathcal{I}$, and the right hand is $\langle \Delta h_1, \mathcal{I}^{-1}\tilde{B}^*\mathcal{I}h_2\rangle_H$.

If $\Delta B = B^*\Delta$, then (15) appears as

$$\langle \Delta Bh_1, h_2\rangle_H = \langle B^*\Delta h_1, h_2\rangle_H = \langle \Delta h_1, Bh_2\rangle_H = \langle \Delta h_1, \mathcal{I}^{-1}\tilde{B}^*\mathcal{I}h_2\rangle_H,$$

and $B = \mathcal{I}^{-1}\tilde{B}^*\mathcal{I}$ means that $\tilde{B} = \tilde{B}^*$.

Thus we have to verify that $\Delta P_\gamma = P_\gamma^* \Delta,\quad \Delta P_{\bar\gamma} = P_{\bar\gamma}^* \Delta$.

From (10) and (3) we derive

$$P_\gamma = \frac{1}{2}\Gamma_{\gamma\bar\gamma}(I + \mathcal{J}_\gamma)\Gamma_{\gamma\bar\gamma}^* \mathcal{J} = \frac{1}{2}\left[\Gamma_{\gamma\bar\gamma}\Gamma_{\gamma\bar\gamma}^* \mathcal{J} + \Gamma_{\gamma\bar\gamma}\mathcal{J}_\gamma\Gamma_{\gamma,\bar\gamma}^* \mathcal{J}\right] = \frac{1}{2}[I + \mathcal{J}\Delta],$$

so $\Delta P_\gamma = \frac{1}{2}(\Delta + \Delta\mathcal{J}\Delta) = \frac{1}{2}(I + \Delta\mathcal{J})\Delta = P_\gamma^*\Delta$.

Likewise, $\Delta P_{\bar\gamma} = \frac{1}{2}(\Delta - \Delta\mathcal{J}\Delta) = P_{\bar\gamma}^*\Delta$. The proof is complete.

Owing to (12) projections $\tilde{P}_\pm$ can be transformed to

$$\tilde{P}_\pm = \frac{1}{2}\mathcal{I}(I \pm \mathcal{J}\Delta)\mathcal{I}^{-1} = \frac{1}{2}\left(I \pm \Delta^{-\frac{1}{2}}\mathcal{J}\Delta^{\frac{1}{2}}\right)\mathcal{I}^{-1} = \frac{1}{2}\mathcal{I}\Delta^{-\frac{1}{2}}(I \pm \mathcal{J})\Delta^{\frac{1}{2}}\mathcal{I}^{-1} = \mathcal{I}\Delta^{-\frac{1}{2}}P_\pm\Delta^{\frac{1}{2}}\mathcal{I}^{-1}.$$

Now denoting $\mathcal{I}_\Delta = \mathcal{I}\Delta^{-\frac{1}{2}}$ we have

$$\tilde{P}_\pm = \mathcal{I}_\Delta P_\pm \mathcal{I}_\Delta^{-1}, \quad \tilde{\mathcal{J}} = \tilde{P}_+ - \tilde{P}_- = \mathcal{I}_\Delta \mathcal{J} \mathcal{I}_\Delta^{-1}. \tag{16}$$

Proposition 3.4 *Let* $\mathcal{I} : H \to \tilde{H}$ *be the inclusion operator of* H *onto* $\tilde{H}$. *Then* $\mathcal{I}_\Delta = \mathcal{I}\Delta^{-\frac{1}{2}} \in [H, \tilde{H}]$ *is an isometry.*

Proof. From (13) one has

$$\langle \mathcal{I}h_1, \tilde{h}_2 \rangle_{\tilde{H}} = \langle \Delta h_1, \mathcal{I}^{-1}h_2 \rangle_H = \langle h_1, \Delta \mathcal{I}^{-1}h_2 \rangle_H,$$

hence $\mathcal{I}^* = \Delta\mathcal{I}^{-1}$, so

$$\langle \mathcal{I}_\Delta h_1, \mathcal{I}_\Delta h_2 \rangle_{\tilde{H}} = \langle \mathcal{I}\Delta^{-\frac{1}{2}}h_1, \mathcal{I}\Delta^{-\frac{1}{2}}h_2 \rangle_{\tilde{H}} = \langle \Delta^{-\frac{1}{2}}h_1, \mathcal{I}^*\mathcal{I}\Delta^{-\frac{1}{2}}h_2 \rangle_H = \langle \Delta^{-\frac{1}{2}}h_1, \Delta^{\frac{1}{2}}h_2 \rangle_H = \langle h_1, h_2 \rangle_H,$$

and the proof is complete.

Note that from the above proof it follows that $\mathcal{I}^*\mathcal{I} = \Delta$, $\mathcal{I}\mathcal{I}^* = \tilde{\Delta}$, since $\mathcal{I}^* = \Delta\mathcal{I}^{-1} = \mathcal{I}^{-1}\tilde{\Delta}$.

Lastly, let $\mathfrak{H}_1$, $\mathfrak{H}_2$ be Hilbert spaces, and an operator $U \in [\mathfrak{H}_1, \mathfrak{H}_2]$ be a partial isometry with the initial space $\mathfrak{N}_1$ and final space $\mathfrak{N}_2$. It is known (see [10; 1.2]) that

1) U^* is a partial isometry with the initial space $\mathfrak{N}_2$ and final space $\mathfrak{N}_1$.

2) U^*U, UU^* are projections in $\mathfrak{H}_1$, $\mathfrak{H}_2$ on $\mathfrak{N}_1$, $\mathfrak{N}_2$ respectively.

We are now ready to prove the main assertion.

Theorem 3.5 *Let* T *be a closed, densely defined Hermitian operator, and let* γ *(*$\mathrm{Im}\,\gamma > 0$*) be arbitrary.*

Let $\{\Gamma, H\} = \{\Gamma_\pm, H_\pm\}$ *be arbitrary reduction operator for* T^*. *Then:*

a) *There exists a reduction operator* $\{\tilde{\Gamma}, \tilde{H}\} = \{\tilde{\Gamma}_{\pm}, \tilde{H}_{\pm}\}$ *such that*

$$\tilde{\Gamma}_{+}\mathfrak{N}_{\gamma} = \tilde{H}_{+}, \quad \tilde{\Gamma}_{+}\mathfrak{N}_{\bar{\gamma}} = \{0\}, \quad \tilde{\Gamma}_{-}\mathfrak{N}_{\gamma} = \{0\}, \quad \tilde{\Gamma}_{-}\mathfrak{N}_{\bar{\gamma}} = \tilde{H}_{-}, \tag{17}$$

where $\mathfrak{N}_{\gamma}$, $\mathfrak{N}_{\bar{\gamma}}$ *are defect subspaces of* T.

b) *The operator* $\tilde{\Gamma}$ *is a partial isometry with the initial subspace* $\mathfrak{N}_{\gamma\bar{\gamma}} = \mathfrak{N}_{\gamma} \oplus \mathfrak{N}_{\bar{\gamma}}$, *and the final subspace* $\tilde{H}$.

Proof. The part a) is already proved, if $\tilde{H}$ be defined by (11), (13), and an isometry of H onto $\tilde{H}$ be given by the operator $\mathcal{I}_{\Delta}$ of Proposition 3.4. If $\tilde{\Gamma} = \mathcal{I}_{\Delta}\Gamma$, then $\tilde{\Gamma}_{\pm} = \tilde{P}_{\pm}\tilde{\Gamma}$, $\tilde{H}_{\pm} = \tilde{P}_{\pm}\tilde{H}$ with orthogonal projections $\tilde{P}_{\pm}$ of formula (16) or (14). Conditions (17) follow from (14) and Propositions 3.2 and 3.3.

To prove the part b) consider the operator $\tilde{\Gamma}_{\gamma\bar{\gamma}} = \tilde{\Gamma}|\mathfrak{N}_{\gamma\bar{\gamma}}$. From formula (5) and conditions (17) it follows that a matrix representation of $\tilde{\Gamma}_{\gamma\bar{\gamma}}$ is of block diagonal form

$$\tilde{\Gamma}_{\gamma\bar{\gamma}} = \begin{bmatrix} \tilde{\theta}_{+}(\gamma) & 0 \\ 0 & \tilde{\theta}_{+}(\bar{\gamma}) \end{bmatrix}, \tag{18}$$

hence the properties (4) and (8) mean that

$$\tilde{\Gamma}^{*}_{\gamma\bar{\gamma}}\tilde{\Gamma}_{\gamma\bar{\gamma}} = I_{\mathfrak{N}_{\gamma\bar{\gamma}}}, \quad \tilde{\Gamma}_{\gamma\bar{\gamma}}\tilde{\Gamma}^{*}_{\gamma\bar{\gamma}} = I_{\tilde{H}}.$$

Obviously $\mathfrak{N}_{\gamma\bar{\gamma}} = (Ker\tilde{\Gamma})^{\perp}$. The proof is complete.

Observe that the operators $\tilde{\theta}_{+}(\gamma) \in [\mathfrak{N}_{\gamma}; \tilde{H}_{+}]$, $\tilde{\theta}_{+}(\bar{\gamma}) \in [\mathfrak{N}_{\bar{\gamma}}; \tilde{H}_{-}]$ of (18) are isometries too.

The reduction operator $\{\tilde{\Gamma}, \tilde{H}\}$ can be referred to as a canonical reduction operator for T^{*} because of the following immediate corollary of Theorem 3.5.

Theorem 3.6 *Let* $\{\tilde{\Gamma}, \tilde{H}\} = \{\tilde{\Gamma}_{\pm}, \tilde{H}_{\pm}\}$ *be a canonical reduction operator for* T^{*}. *Then*

$$\tilde{\Gamma}^{*}\tilde{\Gamma} = \mathcal{P}_{\gamma\bar{\gamma}}, \quad \tilde{\Gamma}^{*}_{+}\tilde{\Gamma}_{+} = \mathcal{P}_{\gamma}, \quad \tilde{\Gamma}^{*}_{-}\tilde{\Gamma}_{-} = \mathcal{P}_{\bar{\gamma}}, \tag{19}$$

where orthogonal projections $\mathcal{P}_{\gamma\bar{\gamma}}$, $\mathcal{P}_{\gamma}$, $\mathcal{P}_{\bar{\gamma}}$ *on* $\mathfrak{N}_{\gamma\bar{\gamma}}$, $\mathfrak{N}_{\gamma}$, $\mathfrak{N}_{\bar{\gamma}}$ *respectively, are defined by the orthogonal decomposition*

$$\mathfrak{D}_{\gamma} = \mathcal{D}(T) \oplus \mathfrak{N}_{\gamma} \ominus \mathfrak{N}_{\bar{\gamma}}.$$

The proof follows from the property 2) of a partial isometry, since from Theorem 3.5 we have

$$Ker\,\tilde{\Gamma}_{+} = \mathcal{D}(T) \oplus \mathfrak{N}_{\bar{\gamma}}, \quad Ker\,\tilde{\Gamma}_{-} = \mathcal{D}(T) \oplus \mathfrak{N}_{\gamma}.$$

Theorem 3.6 causes us be back to formula (8), which corresponds to $\gamma = i$. For arbitrary γ with $\text{Im}\,\gamma = \beta > 0$ one has

$$\langle T^{*}f, g\rangle - \langle f, T^{*}g\rangle = i\beta\left[\langle f_{\gamma}, g_{\gamma}\rangle_{\gamma} - \langle f_{\bar{\gamma}}, g_{\bar{\gamma}}\rangle_{\gamma}\right] = i\beta\left[\langle \mathcal{P}_{\gamma}f, \mathcal{P}_{\gamma}g\rangle_{\gamma} - \langle \mathcal{P}_{\bar{\gamma}}f, \mathcal{P}_{\bar{\gamma}}g\rangle_{\gamma}\right],$$

hence $\{\mathcal{P}_{\gamma\bar{\gamma}}, \mathfrak{N}_{\gamma\bar{\gamma}}\} = \{\mathcal{P}_{\gamma}, \mathcal{P}_{\bar{\gamma}}, \mathfrak{N}_{\gamma}, \mathfrak{N}_{\bar{\gamma}}\}$ is a reduction operator for T^{*}.

The following observation is a variant of Theorem 3.6.

Theorem 3.7 *Let $\{\tilde{\Gamma}, \tilde{H}\} = \{\tilde{\Gamma}_{\pm}, \tilde{H}_{\pm}\}$ be a canonical reduction operator for T^*. Then the isometry $\tilde{\Gamma}^*_{\gamma\bar{\gamma}}$ transfers it to the reduction operator $\{\mathcal{P}_{\gamma\bar{\gamma}}, \mathfrak{N}_{\gamma\bar{\gamma}}\} = \{\mathcal{P}_{\gamma}, \mathcal{P}_{\bar{\gamma}}, \mathfrak{N}_{\gamma}, \mathfrak{N}_{\bar{\gamma}}\}.$*

The proof is covered by (19), since

$$\tilde{\Gamma}^*_{\gamma\bar{\gamma}}\tilde{\Gamma}\mathfrak{D}_{\gamma} = \tilde{\Gamma}^*_{\gamma\bar{\gamma}}\tilde{\Gamma}|\mathfrak{N}_{\gamma\bar{\gamma}} = \tilde{\Gamma}^*_{\gamma\bar{\gamma}}\tilde{\Gamma}_{\gamma\bar{\gamma}}\mathfrak{N}_{\gamma\bar{\gamma}} = \mathfrak{N}_{\gamma\bar{\gamma}} = \mathcal{P}_{\gamma\bar{\gamma}}\mathfrak{D}_{\gamma}.$$

Now let us compare self-adjoint extensions of T, defined by these two reduction operators.

Recall formulas (13), (19), so $P_{-\tilde{V}} = \frac{1}{2}(I_{\tilde{H}} - \tilde{\mathcal{V}})$, where

$$\tilde{\mathcal{V}} = \begin{bmatrix} 0 & \tilde{V}^* \\ \tilde{V} & 0 \end{bmatrix}, \quad (\tilde{V} \in [\tilde{H}_+; \tilde{H}_-] - \text{isometry})$$

is a unitary operator in $[\tilde{H}]$. Then $\tilde{\Gamma}^*_{\gamma\bar{\gamma}}P_{-\tilde{V}}\tilde{\Gamma}_{\gamma\bar{\gamma}} = \frac{1}{2}(I_{\gamma\bar{\gamma}} - \mathcal{V}) = \mathcal{P}_{-V}$, where

$$\mathcal{V} = \begin{bmatrix} 0 & V^* \\ V & 0 \end{bmatrix}, \quad (V = \tilde{\theta}^*_+(\bar{\gamma})\tilde{V}\tilde{\theta}_+(\gamma) \in [\mathfrak{N}_{\gamma}, \mathfrak{N}_{\bar{\gamma}}])$$

is a unitary operator in $[\mathfrak{N}_{\gamma\bar{\gamma}}]$.

In view of

$$P_{-\tilde{V}}\tilde{\Gamma}f = \tilde{\Gamma}_{\gamma\bar{\gamma}}\mathcal{P}_{-V}\tilde{\Gamma}^*_{\gamma\bar{\gamma}}\tilde{\Gamma}f = \tilde{\Gamma}_{\gamma\bar{\gamma}}\mathcal{P}_{-V}\mathcal{P}_{\gamma\bar{\gamma}}f,$$

from the boundary condition $P_{-\tilde{V}}\tilde{\Gamma}f = 0$, defining self-adjoint extension $T_{\tilde{V}}$ of T by means of $\tilde{\Gamma}$, it follows that also $\mathcal{P}_{-V}\mathcal{P}_{\gamma\bar{\gamma}}f = 0$, since $\tilde{\Gamma}_{\gamma\bar{\gamma}}$ is bounded invertible. The last condition, apparently, is von Neumann formula

$$\mathcal{D}(T_V) = \{f \in \mathcal{D}(T^*); \ f = f_0 + V^*f_{\bar{\gamma}} + f_{\bar{\gamma}}, \ f_0 \in \mathcal{D}(T), \ f_{\bar{\gamma}} \in \mathfrak{N}_{\bar{\gamma}}\},$$

defining the domain of self-adjoint extension T_V, hence $\ T_{\tilde{V}} = T_V$.

4   The case of canonical differential operator

4.1. A brief on canonical differential operator in the form presented hereafter one can find in [8], and references there are for details. Let H be a Hilbert space endowed also with an indefinite inner product as at the end of p. 2.2, so we shall use the same notations, and assume that $\dim H_{\pm} = n \leqslant \infty$. Let $L_2(R_+, H)$ be a Hilbert space of H-valued functions, measurable on $R_+ = [0, \infty)$, square norm integrable, and with the inner product

$$\langle f, g \rangle_L = \int_0^{\infty} \langle f(r), g(r) \rangle_H \, dr.$$

Canonical differential expression is of the form

$$k[f] := -i\mathcal{J}f'(r) + V(r)f(r), \quad r \in R_+,$$

where the operator-valued function $V(r) \in L_1(R_+; [H])$ has the properties $V^*(r) = V(r)$, $\mathcal{J}V(r) = -V(r)\mathcal{J}$.

Let $AC(R_+; H)$ be a linear space of functions continuous on R_+, and absolutely continuous on any interval $[0, r)$, $r < \infty$. The expression $k[f]$ defines the minimal symmetric operator C_0 in $L_2(R_+; H)$ with the domain $\mathcal{D}(C_0)$, compiled functions $f \in L_2(R_+; H) \cap AC(R_+; H)$, vanishing at infinity, such that $f(0) = 0$, and $k[f] \in L_2(R_+; H)$.

Let C be the closure of C_0. Then the operator $C^* = C_0^*$ is defined by the formula

$$C^* f = k[f]; \quad \mathcal{D}(C^*) = \{f \in L_2(R_+; H) \cap AC(R_+; H); \ k[f] \in L_2(R_+; H)\}.$$

Properties of function $V(r)$ yield the Lagrange identity

$$\langle C^* f, g \rangle_L - \langle f, C^* g \rangle_L = \int_0^\infty \left[\langle f'(r), i\mathcal{J}g(r) \rangle_H + \langle f(r), i\mathcal{J}g'(r) \rangle_H \right] dr =$$

$$= -i \left[\lim_{r \to \infty} \langle f(r), \mathcal{J}g(r) \rangle_H - \langle f(0), \mathcal{J}g(0) \rangle_H \right] = i\langle \mathcal{J}f(0), g(0) \rangle_H,$$

since existing limit of scalar summable function $\langle f(r), \mathcal{J}g(r) \rangle_H$ must be zero.

Present it in the form

$$\langle C^* f, g \rangle_L - \langle f, C^* g \rangle_L = i\langle \mathcal{J}\Gamma f, \Gamma g \rangle_H, \tag{1}$$

where the operator $\Gamma f = f(0)$, which maps $\mathcal{D}(C^*)$ on H, is a reduction operator $[\Gamma; H]$ for C^*.

4.2. Consider the canonical differential equation

$$-i\mathcal{J}X'(r, \lambda) + V(r)X(r, \lambda) = \lambda X(r, \lambda), \quad r \in R_+ \tag{2}$$

with arbitrary complex parameter λ. Let $\mathcal{E}(r, \lambda)$ be its operator valued solution, satisfying the condition $\mathcal{E}(0, \lambda) = I$.

For $\operatorname{Im} \lambda > 0$, $\operatorname{Im} \zeta < 0$ equation (2) has also solutions

$$\mathcal{A}_+(r, \lambda) = e^{i\lambda r} P_+ + \int_r^\infty e^{i\lambda t} K_r(t) P_+ dt,$$

$$\mathcal{A}_-(r, \zeta) = e^{-i\zeta r} P_- + \int_r^\infty e^{-i\zeta t} K_r(t) P_- dt, \quad K_r(t) \in L_1([r, \infty); [H]) \tag{3}$$

such that $\mathcal{A}_+(r, \lambda)h_+ \in L_2(R_+; H)$, $\mathcal{A}_-(r, \zeta)h_- \in L_2(R_+; H)$, $h_\pm \in H_\pm$.

Introducing operator functions

$$\mathcal{A}_+(\lambda) := \mathcal{A}_+(0, \lambda) = P_+ + \int_0^\infty e^{i\lambda t} K_0(t) P_+ dt, \quad \mathcal{A}_+(\lambda) \in [H],$$

$$\mathcal{A}_-(\zeta) := \mathcal{A}_+(0, \zeta) = P_- + \int_0^\infty e^{-i\zeta t} K_0(t) P_- dt, \quad \mathcal{A}_-(\zeta) \in [H],$$

we have $\mathcal{A}_+(r,\lambda) = \mathcal{E}(r,\lambda)\mathcal{A}_+(\lambda)$, $\mathcal{A}_-(r,\zeta) = \mathcal{E}(r,\zeta)\mathcal{A}_-(\zeta)$.

Thus we get the following description of defect subspaces of C

$$\mathcal{N}_\lambda = \{h_\lambda(r) = \mathcal{E}(r,\lambda)\mathcal{A}_+(\lambda)h_+,\ h_+ \in H_+\},\ \mathcal{N}_\zeta = \{h_\zeta(r) = \mathcal{E}(r,\zeta)\mathcal{A}_-(\zeta)h_-,\ h_- \in H_-\},$$

hence in the decomposition $H = H_+ \oplus H_-$ we have

$$\Gamma\mathcal{N}_\lambda = \left\{\mathcal{A}_+(\lambda)h_+ = \begin{bmatrix} A_{11}(\lambda)h_+ \\ A_{21}(\lambda)h_+ \end{bmatrix}\right\},\ \Gamma\mathcal{N}_\zeta = \left\{\mathcal{A}_-(\zeta)h_- = \begin{bmatrix} A_{12}(\zeta)h_- \\ A_{22}(\zeta)h_- \end{bmatrix}\right\}. \tag{4}$$

Now let $\lambda = \mu + i\nu$, $\nu > 0$. With the help of a standard procedure, which uses equation (2) and its adjoint equation, one can see that

$$\left[\mathcal{A}_+^*(r,\lambda)\mathcal{J}\mathcal{A}_+(r,\lambda)\right]' = i(\lambda - \bar{\lambda})\mathcal{A}_+^*(r,\lambda)\mathcal{A}_+(r,\lambda) = -2\nu\mathcal{A}_+^*(r,\lambda)\mathcal{A}_+(r,\lambda), \tag{5}$$

as well as

$$\left[\mathcal{A}_-^*(r,\bar{\lambda})\mathcal{J}\mathcal{A}_+(r,\lambda)\right]' = 0. \tag{6}$$

Integration of (5) on $[0,r]$ yields

$$\mathcal{A}_+^*(r,\lambda)\mathcal{J}\mathcal{A}_+(r,\lambda) - \mathcal{A}_+^*(\lambda)\mathcal{J}\mathcal{A}_+(\lambda) = -2\nu\int_0^r \mathcal{A}_+^*(t,\lambda)\mathcal{A}_+(t,\lambda)\,dt\,,$$

and, on account of $\lim\limits_{r\to\infty}\mathcal{A}_+(r,\lambda) = 0$ which follows from (3), we get

$$\mathcal{A}_+^*(\lambda)\mathcal{J}\mathcal{A}_+(\lambda) = 2\nu\int_0^\infty \mathcal{A}_+^*(t,\lambda)\mathcal{A}_+(t,\lambda)\,dt > 0\,. \tag{7}$$

Equation (6) means that $\mathcal{A}_-^*(r,\bar{\lambda})\mathcal{J}\mathcal{A}_+(r,\lambda) = const$, hence, by the above reason, one has

$$\mathcal{A}_-^*(\bar{\lambda})\mathcal{J}\mathcal{A}_+(\lambda) = 0\,. \tag{8}$$

Similar derivations one can repeat for ζ ($\operatorname{Im}\zeta < 0$) and, setting $\zeta = \bar{\lambda}$, arrive at

$$\mathcal{A}_-^*(\bar{\lambda})\mathcal{J}\mathcal{A}_-(\bar{\lambda}) = -2\nu\int_0^\infty \mathcal{A}_-^*(t,\bar{\lambda})\mathcal{A}_-(t,\lambda)\,dt < 0\,, \tag{9}$$

$$\mathcal{A}_+^*(\lambda)\mathcal{J}\mathcal{A}_-(\bar{\lambda}) = 0\,. \tag{10}$$

Denote

$$A_+(\lambda) = 2\nu\int_0^\infty \mathcal{A}_+^*(t,\lambda)\mathcal{A}_+(t,\lambda)\,dt\,,\quad A_-(\bar{\lambda}) = 2\nu\int_0^\infty \mathcal{A}_-^*(t,\bar{\lambda})\mathcal{A}_-(t,\bar{\lambda})\,dt,$$

so $A_+(\lambda) \in [H_+]$, $A_-(\bar{\lambda}) \in [H_-]$. Since they are strictly positive, hence operators $A_+^{\frac{1}{2}}(\lambda)$, $A_-^{\frac{1}{2}}(\bar{\lambda})$ and their inverses exist. Now, referring to formula (4), let us form the operator

$$
\mathcal{A}_\lambda = \begin{bmatrix} A_{11}(\lambda)A_+^{-\frac{1}{2}}(\lambda) & A_{12}(\bar{\lambda})A_-^{-\frac{1}{2}}(\bar{\lambda}) \\ A_{21}(\lambda)A_+^{-\frac{1}{2}}(\lambda) & A_{22}(\bar{\lambda})A_-^{-\frac{1}{2}}(\bar{\lambda}) \end{bmatrix}, \quad \mathcal{A}_\lambda \in [H].
$$

From (7) - (10), it is clear that

$$
\mathcal{A}_\lambda^* \mathcal{J} \mathcal{A}_\lambda = \mathcal{J}, \tag{11}
$$

hence also

$$
\mathcal{A}_\lambda^{-1} = \mathcal{J}\mathcal{A}_\lambda^*\mathcal{J}, \quad \mathcal{A}_\lambda\mathcal{J}\mathcal{A}_\lambda^* = \mathcal{J}.
$$

Obviously $A_+^{-\frac{1}{2}}(\lambda)H_+ = H_+$, $A_-^{-\frac{1}{2}}(\bar{\lambda})H_- = H_-$, so formula (4) can be presented as

$$
\Gamma\mathcal{N}_\lambda = \mathcal{A}_\lambda H_+ = \mathcal{A}_\lambda P_+ H, \quad \Gamma\mathcal{N}_{\bar{\lambda}} = \mathcal{A}_\lambda H_- = \mathcal{A}_\lambda P_- H,
$$

and from (11) it is immediate that

$$
P_\lambda = \mathcal{A}_\lambda P_+ \mathcal{A}_\lambda^* \mathcal{J}, \quad P_{\bar{\lambda}} = -\mathcal{A}_\lambda P_- \mathcal{A}_\lambda^* \mathcal{J}
$$

are disjoint projections on $\Gamma\mathcal{N}_\lambda$, $\Gamma\mathcal{N}_{\bar{\lambda}}$ respectively.

We regard $\mathcal{A}_\lambda$ as the analog of $\Gamma_{\lambda\bar{\lambda}}$ in p. 3.2 for the case of the symmetric differential operator C. Now, following and p. 3.3, we introduce the Hilbert space $\tilde{H}$ and inclusion operator $\mathcal{I} : H \to \tilde{H}$ as

$$
\langle \tilde{h}_1, \tilde{h}_2 \rangle_{\tilde{H}} = \langle \mathcal{I}h_1, \mathcal{I}h_2 \rangle_{\tilde{H}} = \langle \Delta_\lambda h_1, h_2 \rangle_H, \quad \Delta_\lambda = \mathcal{J}\mathcal{A}_\lambda\mathcal{A}_\lambda^*\mathcal{J},
$$

Derivations similar to those in p. 3.3 lead to

$$
\tilde{H} = \tilde{H}_+ \oplus \tilde{H}_- = \mathcal{I}\Gamma\mathcal{N}_\lambda \oplus \mathcal{I}\Gamma\mathcal{N}_{\bar{\lambda}}, \quad \tilde{H}_\pm = \tilde{P}_\pm \tilde{H},
$$

where

$$
\tilde{P}_\pm = \pm\mathcal{I}\mathcal{A}_\lambda P_\pm \mathcal{A}_\lambda^* \mathcal{J}\mathcal{I}^{-1} = \frac{1}{2}\mathcal{I}(I \pm \mathcal{J}\Delta_\lambda)\mathcal{I}^{-1} = \mathcal{I}_\Delta P_\pm \mathcal{I}_\Delta^{-1},
$$

and $\mathcal{I}_\Delta = \mathcal{I}\Delta_\lambda^{-\frac{1}{2}} \in [H, \tilde{H}]$ is an isometry.

The operator $\tilde{\Gamma} = \mathcal{I}_\Delta\Gamma : \mathcal{D}(C^*) \to \tilde{H}$ is such that

$$
\langle \tilde{\mathcal{J}}\tilde{\Gamma}f, \tilde{\Gamma}g \rangle_{\tilde{H}} = \langle \mathcal{J}\Gamma f, \Gamma g \rangle_H, \quad \tilde{\mathcal{J}} = \mathcal{I}_\Delta \mathcal{J} \mathcal{I}_\Delta^{-1}.
$$

Thus $\{\tilde{\Gamma}, \tilde{H}\} = \{\tilde{\Gamma}_\pm, \tilde{H}_\pm\}$ is the reduction operator for C^* with the properties of canonical reduction operator

$$
\tilde{\Gamma}_+\mathcal{N}_\lambda = \tilde{H}_+, \quad \tilde{\Gamma}_+\mathcal{N}_{\bar{\lambda}} = \{0\}, \quad \tilde{\Gamma}_-\mathcal{N}_\lambda = \{0\}, \quad \tilde{\Gamma}_-\mathcal{N}_{\bar{\lambda}} = \tilde{H}_-.
$$

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